RESEARCH STATEMENT

JENNIFER FROELICH

1. Motivation

Deformation theory has to do with the behavior of mathematical objects, such as group representations, under small perturbations. This theory is useful in both pure and applied mathematics and has led to the solution of many long-standing problems. In particular, in number theory, Wiles and Taylor used universal deformation rings of Galois representations in the proof of Fermat's Last Theorem. The main motivation for determining universal deformation rings for finite groups is to test or verify conjectures about the ring structure of universal deformation rings for arbitrary Galois groups.

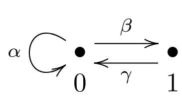
2. Goals of Research Related to Thesis

Consider the symmetric group S_5 and its non-trivial double cover $\tilde{S}_5$ with generalized quaternion Sylow 2-subgroups. Let k be an algebraically closed field of characteristic 2 and let $B_0(kS_5)$ be the principal block of the group algebra kS_5 where the principal block is the block in which the trivial kS_5-module k lies, see [1]. The goal of my current research is to find all $B_0(kS_5)$-modules with stable endomorphism ring isomorphic to k that also have stable endomorphism ring isomorphic to k as $k\tilde{S}_5$-modules and then to calculate the universal deformation ring for each of these modules.

3. Methods Used in Research Related to Thesis

Let k be an algebraically closed field of characteristic 2. It can be shown that every finite dimensional algebra over k is Morita equivalent to kQ/I for some directed graph Q and some ideal I of kQ generated by relations, see [2]. Here kQ denotes the path algebra of Q whose underlying k-vector space is generated by all paths in Q.

Let Λ be the finite dimensional k-algebra with directed graph

$Q =$ 

and relations $\alpha^2 = c(\gamma\beta\alpha)^2$, $(\gamma\beta\alpha)^2 = (\alpha\gamma\beta)^2$, and $\beta\gamma = 0$. With this description, we have that Λ is Morita equivalent to the principal block $B_0(kS_5)$ of the group algebra kS_5 for a suitable $c = 0$ or 1, see [5]. There are two simple Λ-modules up to isomorphism corresponding to the vertices 0 and 1 which we denote by S_0 and S_1, and there are two projective Λ-modules up to isomorphism.

Let $\tilde{\Lambda}$ be the finite dimensional k-algebra with directed graph Q as above and relations $\gamma\beta\gamma = \alpha\gamma(\beta\alpha\gamma)^3, \alpha^2 = \gamma\beta(\alpha\gamma\beta)^3 + c(\alpha\gamma\beta)^3, \beta\gamma\beta = \beta\alpha(\gamma\beta\alpha)^3$, and $\beta\alpha^2 = 0$. With this description, we have that $\tilde{\Lambda}$ is Morita equivalent to the principal block $B_0(k\tilde{S}_5)$ of the group algebra $k\tilde{S}_5$ for a suitable $c \in k$, see [5]. Since Λ is isomorphic to a quotient algebra of $\tilde{\Lambda}$, the inflation of S_0 and S_1 to $\tilde{\Lambda}$ represents the two isomorphism classes of simple $\tilde{\Lambda}$-modules. We denote these inflations also by S_0 and S_1.

Let $A = \Lambda$ or $\tilde{\Lambda}$. Let M and N be A-modules and let $P(M,N)$ is the subspace of $Hom_A(M,N)$ consisting of the maps which factor through a projective module. Then $\underline{Hom}_A(M,N)$ is the quotient $Hom_A(M,N)/P(M,N)$. $\underline{End}_A(M) = \underline{Hom}_A(M,M)$ is called the *stable endomorphism ring* of M.

Given the directed graph Q described above, a *string* S is a sequence of arrows and formal inverses of arrows such that no subsequence or its inverse equals zero by the relations for Q. To each such string S, one can associate an A-module $M(S)$. $M(S)$ is called the *string module with underlying string* S. Consider for example the string $S = \alpha\gamma$. To this string, associate the A-module $M(S)$ where $M(S)$ is the uniserial

A-module with composition factors S_1, S_0, and S_0. The notation $M(S) = \begin{smallmatrix}1\\0\\0\end{smallmatrix}$ is used to describe this. All A-modules $M(S)$ with underlying string S can be described in a similar way.

The objective of the first part of my work is to find all Λ-modules $M(S)$ with underlying string S and with stable endomorphism ring isomorphic to k that also have stable endomorphism ring isomorphic to k as $\tilde{\Lambda}$-modules. The reason is that if $\underline{End}_A(M(S)) \cong k$, then $M(S)$ has a well-defined universal deformation ring which is a complete local commutative Noetherian ring with residue field k.

Let M be a non-projective indecomposable Λ-module, which is viewed as a $\tilde{\Lambda}$-module by inflation. Then I prove $\underline{End}_{\tilde{\Lambda}}(M) \cong k$ if and only if $End_\Lambda(M) \cong k$. Moreover, $End_\Lambda(M(S)) \cong k$ if and only if $M(S) \in \{S_0, S_1, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}0\\0\\1\end{smallmatrix}$, and $\begin{smallmatrix}1\\0\\0\end{smallmatrix}\}$.

Next, I calculate $Ext^1_\Lambda(M,M)$ and $Ext^1_{\tilde{\Lambda}}(M,M)$ for the above modules, and I find that $Ext^1_\Lambda(M,M) = Ext^1_{\tilde{\Lambda}}(M,M) = 0$ for $M = S_1$ and $Ext^1_\Lambda(M,M) \cong Ext^1_{\tilde{\Lambda}}(M,M) \cong k$ for $M = S_0, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}0\\0\\1\end{smallmatrix}$, and $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$. These calculations provide restrictions on what the universal deformation ring for each module can be.

Let $G = S_5$ or $\tilde{S}_5$ and let M be a finitely generated kG-module. Let $\mathcal{C}$ be the category of all complete local commutative Noetherian rings with residue field k. Let $R \in \mathcal{C}$. A *lift* of M over R is a finitely generated RG-module V which is free over R such that $k \otimes_R V \cong M$ as kG-modules. A *deformation* of M over R is an isomorphism class of a lift V of M over R. The *deformation functor* is the covariant functor $F : \mathcal{C} \to Sets$ which sends $R \in \mathcal{C}$ to the set of deformations of M over R. In case there is a ring $R(G,M) \in \mathcal{C}$ which represents the functor F, i.e. $F \cong Hom_{\mathcal{C}}(R(G,M), -)$, we call $R(G,M)$ the *universal deformation ring* of M. This deformation theory was introduced by Mazur, using work of Schlessinger, in [6]. It follows from [3] that M has a universal deformation ring $R(G,M)$ in case M has stable endomorphism ring isomorphic to k. The universal deformation rings modulo 2 are defined as $R(G,M)/2R(G,M)$.

Since $Ext^1_\Lambda(M,M) = Ext^1_{\tilde{\Lambda}}(M,M) = 0$ for $M = S_1$, the universal deformation ring modulo 2 for $M = S_1$ must be isomorphic to k. Also since $Ext^1_\Lambda(M,M) \cong Ext^1_{\tilde{\Lambda}}(M,M) \cong k$ for $M = S_0, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}0\\0\\1\end{smallmatrix}$, and $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$, the universal deformation rings modulo 2 for these modules must be isomorphic to a quotient of the formal power series ring $k[[t]]$. I prove the following: $R(S_5,M)/2R(S_5,M) \cong k[t]/(t^2)$ if $M = S_0, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}0\\0\\1\end{smallmatrix}$, or $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$. Also, $R(\tilde{S}_5,M)/2R(\tilde{S}_5,M) \cong k[t]/(t^2)$ if $M = S_0, \begin{smallmatrix}0\\1\end{smallmatrix}$, or $\begin{smallmatrix}1\\0\end{smallmatrix}$ and $R(\tilde{S}_5,M)/2R(\tilde{S}_5,M) \cong k[t]/(t^3)$ if $M = \begin{smallmatrix}0\\0\\1\end{smallmatrix}$ or $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$.

In the future, I will calculate the full universal deformation ring for all the above modules using techniques of ordinary representation theory.

4. Other Research

Very recently, my work has led to the calculation of the deformation rings associated to the two-torsion on some elliptic curves E over $\mathbb{Q}$. This is joint work with F. Bleher and T. Chinburg. By replacing the group S_5 by S_4, we consider the non-trivial double cover $\tilde{S}_4$ with generalized quaternion Sylow 2-subgroups. Let V be a 2-dimensional irreducible representation of S_4 over $\mathbb{Z}_2$, which is unique up to isomorphism. We can compute the universal deformation ring $R(\tilde{S}_4, V)$ of the inflation of V to $\tilde{S}_4$. The representation V was studied in [4], where it was shown to arise from S_4-extensions of $\mathbb{Q}$ which are unramified outside of p. We can show that under some additional mild hypotheses on p, which are satisfied by many p, the universal deformation ring $R(G_{(p)}, V)$ of V as a representation of the Galois group $G_{(p)}$ of the maximal unramified outside p extension of $\mathbb{Q}$ is equal to $R(\tilde{S}_4, V)$ and is a local complete intersection ring. It was noted by Bleher and Chinburg in [4] that the deformation ring $R(S_4, V)$ is not a local complete intersection ring, but this ring was not sufficient to compute $R(G_{(p)}, V)$. The interest of the above result is that by moving to a double cover $\tilde{S}_4$ one does arrive at a computable deformation ring $R(\tilde{S}_4, V)$ which equals $R(G_{(p)}, V)$.

Elliptic curves enter into this topic because V arises as the Galois representation associated to the 2-torsion $E[2]$ of suitable elliptic curves E over $\mathbb{Q}$. The deformation ring $R(G_{(p)}, V)$ is of interest in the theory of modular forms. More explicitly, that theory involves $R(G'_{(p)}, V)$ and related deformation rings with additional deformation conditions, where $G'_{(p)}$ is the Galois group of the maximal extension of $\mathbb{Q}$ which is unramified outside 2, p, and ∞ and the other places of $\mathbb{Q}$ where E has bad reduction. We are now looking into the problem of constructing E for a given p which have good reduction outside p.

5. Future Research

After completing the above described research, I plan to consider similar questions for other group algebras. First, I would like to consider the non-trivial double cover $\hat{S}_5$ of S_5 with semi-dihedral Sylow 2-subgroups. Next, Erdmann has classified all blocks of group algebras with dihedral, semi-dihedral, and quaternion defect groups. The principal blocks of S_5, $\tilde{S}_5$, and $\hat{S}_5$ considered above are the smallest examples (with respect to the order of the defect groups) of whole series of blocks described by Erdmann. All these examples are based on the same quiver Q but with different relations. I plan to determine the universal deformation rings for all blocks belonging to these series.

I also want to look into whether these universal deformation rings arise from arithmetic as was shown to be the case for S_4 in [4]. In particular, I am interested in computing the universal deformation rings of representations of the maximal algebraic extension of a number field which is unramified outside a given finite set of places. A basic part of this problem is to construct central extensions of a particular group as Galois groups over the base field. There is a large literature on this topic, see for example the work of J.F. Mestre. This literature connects the problem to Stiefel-Whitney classes of orthogonal representations and to Hasse-Witt invariants of trace forms.

References

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