Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

Professors Frauke Bleher and Muthu Krishnamurthy

August 18, 2017, 9:00 a.m. - 12:00 p.m. in 221 MLH

Instructions:

- Do EXACTLY TWO problems from EACH of the four sections.
- Please start a new page for every new problem and put your name on each sheet.
- Justify your answers and show your work.
- Please write legibly.
- In answering any part of a question, you may assume the results in previous parts of the SAME question, even if you have not solved them.
- Please turn in the exam questions with your solutions.

Notations:

We adopt standard notations. Namely:

- We write $\mathbb{C}, \mathbb{R}$ and $\mathbb{Q}$ to denote the field of complex numbers, real numbers and rational numbers, respectively; we write $\mathbb{Z}$ to denote the ring of rational integers.
- Throughout this exam, R denotes a ring with identity $1 \neq 0$; R is called an integral domain if it is commutative with no zero divisors.
- All R-modules are assumed to be unital left R-modules.

1 Groups

1. Let G be a group that acts *transitively* on a *finite* set A. (G is NOT assumed to be finite.) Let H be a normal subgroup of G, and let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the distinct orbits of H on A.

   Prove that the action of G on A induces a well-defined action of G on $\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$, and prove that this action is transitive. Deduce that all orbits of H on A have the same cardinality.

2. Let p be a prime number. If G is a finite group, denote the number of Sylow p-subgroups of G by $n_p(G)$. Suppose A and B are finite groups, and that P is a Sylow p-subgroup of A and Q is a Sylow p-subgroup of B.

   Prove that $P \times Q$ is a Sylow p-subgroup of $A \times B$, and that $n_p(A \times B) = n_p(A)n_p(B)$.

3. Let G be a group with identity element e. Suppose $n > 1$ is a fixed integer such that $(xy)^n = x^n y^n$ for all $x, y \in G$. Define

$$G^{(n)} = \{x^n \mid x \in G\} \quad \text{and} \quad G_{(n)} = \{x \in G \mid x^n = e\}\,.$$

   (a) Show that $G^{(n)}$ and $G_{(n)}$ are normal subgroups of G.

   (b) If G is finite, show that the order of $G^{(n)}$ equals the index of $G_{(n)}$ in G.

   (c) Show that for all $x, y \in G$, we have $x^{1-n}y^{1-n} = (xy)^{1-n}$. Use this to deduce that $x^{n-1}y^n = y^n x^{n-1}$.

2 Rings

1. Let $n \geq 2$ be an integer, and let $\mathrm{Mat}_n(R)$ denote the ring of $n \times n$ matrices with entries in R.

   Prove that every two-sided ideal of $\mathrm{Mat}_n(R)$ is equal to $\mathrm{Mat}_n(J)$ for some two-sided ideal J of R.

2. Let R be an integral domain.

   (a) Prove that every prime element in R is irreducible in R. (Please include definitions of irreducible and prime elements.)

   (b) If R is a Principal Ideal Domain, prove that the converse is also correct.

3. Determine which of the following ideals are prime ideals. Please justify your answers.

   (a) The ideal generated by i in the ring of Gaussian integers $\mathbb{Z}[i]$.

   (b) The ideal generated by y and $x^2 + yx + 1$ in $\mathbb{Z}[x, y]$.

   (c) The ideal generated by $y^2 - x^3 - x^2$ in $\mathbb{C}[x, y]$.

3 Linear Algebra and Module Theory

1. Suppose

$$\begin{array}{ccccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D & \xrightarrow{\delta} & E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & D' & \xrightarrow{\delta'} & E' \end{array}$$

   is a commutative diagram of R-modules with exact rows.

   Prove: If g and k are isomorphisms, f is surjective and i is injective, then h is an isomorphism.

2. Let F be a field and let V be a vector space over F. Let $V^* = \mathrm{Hom}_F(V, F)$ be the dual vector space of V and let $V^{**} = (V^*)^*$ be its double dual. Define a map $\tau : V \to V^{**}$ by $\tau(v)(f) = f(v)$ for all $v \in V$ and $f \in V^*$.

   (a) Prove that τ is an injective linear transformation.

   (b) If V is finite dimensional over F, prove that τ is an isomorphism.

3. A module over a ring R is simple if it has no non-zero proper submodules and it is semi-simple if it is a direct sum of simple modules. One defines a linear operator T on a finite dimensional vector space V (over a field k) to be semi-simple if the corresponding $k[x]$-module (V, T) is semi-simple.

   (a) Describe all simple $k[x]$-modules. Please justify your answer.

   (b) Show that if T is diagonalizable, then it is semi-simple. Show that the converse holds if k is algebraically closed.

4 Field Theory

1. Let p be a prime number, let $f(x) = x^p - 5 \in \mathbb{Q}[x]$, and let G be the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$.

   Prove that G is isomorphic to the group of matrices

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{F}_p, a \neq 0 \right\}.$$

2. Let $F \subseteq K \subseteq L$ be a tower of fields.

   Prove: L/F is an algebraic extension if and only if L/K and K/F are both algebraic extensions.

3. Determine the order of the Galois group of $K/\mathbb{Q}$ where K is the splitting field of $f(x) = x^6 - 4x^3 + 1$ over $\mathbb{Q}$.