Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

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Instructions:

- Do EXACTLY TWO problems from EACH of the four sections.
- Please start a new page for every new problem and put your name on each sheet.
- Justify your answers and show your work.
- Please write legibly.
- In answering any part of a question, you may assume the results in previous parts of the SAME question, even if you have not solved them.
- Please turn in the exam questions with your solutions.

Notations:

We adopt standard notations. Namely:

- We write $\mathbb{C}, \mathbb{R}$ and $\mathbb{Q}$ to denote the field of complex numbers, real numbers and rational numbers, respectively. We write $\mathbb{Z}$ to denote the ring of rational integers. If p is a prime number then $\mathbb{F}_p$ denotes the finite field with p elements.
- Throughout this exam, R denotes a ring with identity $1 \neq 0$; R is called an integral domain if it is commutative with no zero divisors.
- All R-modules are assumed to be unital left R-modules.

1 Groups

1. Prove that every group of order $616 = 2^3 \cdot 7 \cdot 11$ is not simple.

2. Let G be a group and let $H \leq G$. Let $\mathcal{C} = \{gH \mid g \in G\}$ be the set of distinct left cosets of H in G.

   Prove that G acts on $\mathcal{C}$ by left multiplication (i.e. prove this is a group action), show that this action is transitive and that the kernel of this action is equal to $\bigcap_{g \in G} gHg^{-1}$.

3. Let G be a simple group with 360 elements, and let H be a proper subgroup. Let $\mathcal{C}_\ell = \{gH \mid g \in G\}$ be the set of left cosets of H in G, and let $\mathcal{C}_r = \{Hg \mid g \in G\}$ be the set of right cosets of H in G. Suppose $\mathcal{C}_\ell = \mathcal{C}_r$ as subsets of $\mathcal{P}(G)$. Find the order of H.

2 Rings

1. Let R and S be commutative rings, and let $\varphi : R \to S$ be a surjective ring homomorphism. Let P be an ideal of R with $\mathrm{Ker}(\varphi) \subseteq P$.

   Prove that P is a prime ideal of R if and only if $\varphi(P)$ is a prime ideal of S.

2. Let F be a field. Prove that the set R of polynomials $f(x) \in F[x]$ whose coefficient of x is equal to 0 is a subring of $F[x]$ and that R is not a UFD.

3. Let R be a ring generated by invertible elements, equivalently, each element of R can be written as a finite sum of invertible elements.

   Show that 1 can be written as a sum of an even number of invertible elements $1 = u_1 + \ldots + u_{2n}$ if and only if there is no surjective ring homomorphism $f : R \to \mathbb{F}_2$.

3 Linear Algebra and Module Theory

1. Let R be an integral domain, and let M be an R-module. Recall that the rank of M is the maximum number of R-linearly independent elements in M.

   Prove: If M has a submodule N that is a free R-module of rank n and M/N is a torsion R-module, then M has rank n.

2. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

   Find the rational canonical form of A over $\mathbb{Q}$, and find the Jordan canonical form of A over $\mathbb{C}$.

3. Let p be a prime number, and let $A = (a_{ij})_{1 \le i,j \le p} \in M_p(\mathbb{F}_p)$ be the $p \times p$ matrix with $a_{i\,i+1} = 1$ for $i = 1, \ldots, p-1$ and $a_{p,1} = 1$, and $a_{ij} = 0$ for all other values of the pair (i, j). Find the Jordan canonical form of A over $\mathbb{F}_p$.

4 Field Theory

1. Let F be a field, and let $\sigma : F \to L$ be an embedding of F into an algebraically closed field L. Suppose α is algebraic over F.

   Prove that there exists an embedding $\tau : F(\alpha) \to L$ extending σ. Moreover, prove that the number of distinct embeddings $F(\alpha) \to L$ extending σ is equal to the number of distinct roots (in some algebraic closure of F) of the minimal polynomial $m_{\alpha,F}(x)$ of α over F.

2. Find the splitting field K of $f(x) = x^4 + 10x^2 + 5$ over $\mathbb{Q}$. Determine a set of generators for the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ and describe how each of these generators permutes the roots of $f(x)$. Determine the isomorphism type of G.

3. Let $K \subset L$ be a finite field extension such that $K \neq L$. Suppose that for every $d \,\big|\, [L : K]$, there is a *unique* subextension $K \subseteq F \subseteq L$ such that $[F : K] = d$.

   (i) Give an example of such an extension that is Galois, and an example of such an extension that is not Galois.

   (ii) Suppose the extension above is Galois. Show that $\mathrm{Gal}(L/K)$ is abelian.