Qualifying Exam —Analysis
Winter 2017

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Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains a section of 5 problems in real analysis and a section of 5 problems in complex analysis. For maximum points you must submit solutions for 7 problems. Also you must attempt at least 2 problems in each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- You are not allowed to use a cell phone or a calculator during the exam.

Real Analysis

R - I: Solve at your choice ONE of the following problems:

a) Let E be a measurable set such that $0 < \mu(E) < \infty$. Show that the set $E - E = \{x - y : x, y \in E\}$ contains an nonempty open interval.

b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is measurable then the set $\{x \in \mathbb{R} : \mu(f^{-1}(x)) > 0\}$ has measure zero.

R - II: Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesque integrable on the real line. Show that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| d\mu(x) = 0.$$

R - III: Let (X, d) be a compact metric space and let $f : X \to X$ be a function such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Show that f has a unique fixed point.

R - IV: Let $f : [0, 1] \to \mathbb{R}_+$ be an integrable function and let $F_n \subseteq [0, 1]$ be a sequence of measurable sets such that $\int_{F_n} f d\mu \to 0$, as $n \to \infty$. Show that $\mu(F_n) \to 0$, as $n \to \infty$.

R - V: Let $F_k \subset [0, 1]$, $k \in \mathbb{N}$ be measurable sets, and there exists $\delta > 0$ such that $m(F_k) \geq \delta$ for all k. Assume the sequence $a_k \geq 0$ satisfies

$$\sum_{k=1}^{\infty} a_k \chi_{F_k}(x) < \infty \text{ for a.e. } x \in [0, 1].$$

Show that

$$\sum_{k=1}^{\infty} a_k < \infty.$$

Make sure you include all the details in your arguments.

Complex analysis

C - I: Define $D = \{z \in \mathbb{C} : |z| < 1\}$, $D_+ = \{z \in \mathbb{C} : |z - i|^2 < 2\}$, $D_- = \{z \in \mathbb{C} : |z + i|^2 < 2\}$, and let $\Omega = D_+ \cap D_-$. Construct a bi-holomophic map from Ω to D.

C - II: Let $f \in \mathbb{C}[z]$ be a polynomial of degree n. Let $\alpha_1, \ldots, \alpha_n$ be roots of $f(z)$ and let $\beta_1, \ldots, \beta_{n-1}$ be roots of $f'(z)$.

1. If for all $i \in \{1, \ldots, n\}$ we have $Re(\alpha_i) > 0$ then prove that $Re(\beta_j) > 0$ for all $j \in \{1, \ldots, n-1\}$;
2. If for all $i \in \{1, \ldots, n\}$ and $|\alpha_i| < 1$ then prove that $|\beta_j| < 1$ for all $j \in \{1, \ldots, n-1\}$.

C - III: Let $f(z)$ be a non-constant analytic function on $D_2 = \{z \in \mathbb{C} : |z| < 2\}$. If $|f(z)| \equiv 1$ for all z such that $|z| = 1$, then prove that f has at least one zero in $D = \{z \in \mathbb{C} : |z| < 1\}$.

C - IV: Compute the following integral

$$\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^3 \, dx.$$

C - V: Show that all the roots of the equation $e^z = 3z^2$ in $D = \{z \in \mathbb{C} : |z| < 1\}$ are real.