Qualifying Exam —Analysis
Winter 2018

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Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains a section of 4 problems in real analysis and a section of 4 problems in complex analysis. For maximum points you must submit solutions for 6 problems. Also you must attempt at least 2 problems in each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- You are not allowed to use a cell phone or a calculator during the exam.

Real Analysis

R - I: Solve at your choice ONE of the following problems:

a) Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $3\mu(E \cap (a, b)) \leq b - a$ for all $a < b$. Find $\mu(E)$. Make sure you include all the details in your arguments.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Show that f has at most countably many discontinuity points. Conversely, if $A \subset \mathbb{R}$ is a countable subset then there exists a increasing function f whose discontinuity points coincide with A. Make sure you include all the details in your arguments.

R - II: Let $p \geq 1$. Assume that f is an absolute continuous function on any compact interval and moreover $f' \in L^p(\mathbb{R}, \mu)$. Show that

$$\sum_{n \in \mathbb{Z}} |f(n+1) - f(n)|^p < \infty.$$

R - III: Let (X, d) be a compact metric space and let $f : X \to X$ be an isometry (i.e. $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$). Show that f is a homeomorphism (i.e. it is continuous, invertible, and the inverse is continuous as well).

R - IV: Let $f_n \in L^3((0, 1))$ nonnegative functions such that $\|f_n\|_3 = 1$ for all n and $f_n \to 0$ almost everywhere as $n \to \infty$. Show that $\int_0^1 f_n d\mu \to 0$ as $n \to \infty$.

Complex analysis

C - I: Solve at your choice ONE of the following problems:

a) Compute the following integral

$$\int_0^\infty \frac{dx}{1+x^7}.$$

b) Construct a conformal map from the unit disk onto the infinite horizontal strip $|Im(z)| < 1$. Make sure you include all the details in your arguments.

c) TRUE-FALSE: Let f be analytic on the open punctured unit disk $\mathbb{D} \setminus \{0\}$. Can f' have a polar singularity of order one at 0? Make sure you include all the details in your arguments.

C - II: Suppose $f : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is a nonconstant, real-valued harmonic function on the punctured plane. Prove that the image of f is all of $\mathbb{R}$.

C - III: Suppose f is a holomorphic function on $\{z \in \mathbb{C} \,|\, |z| < 1\}$, the open unit disk, with the property that $\text{Re}f(z) > 0$ for every point z in the disk. Prove that $|f'(0)| \leq 2\text{Re}f(0)$.

C - IV: Let $f : \mathbb{C} \to \mathbb{C}$ be an injective holomorphic function. Show there exists $a, b \in \mathbb{C}$ such that $f(z) = az + b$.