Qualifying Exam —Analysis
Summer 2017

Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains a section of 4 problems in real analysis and a section of 4 problems in complex analysis. For maximum points you must submit solutions for 6 problems. Also you must attempt at least 2 problems in each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- You are not allowed to use a cell phone or a calculator during the exam.

Good luck!

Real Analysis

R - I: Solve at your choice ONE of the following problems:

a) Let E be the subset of all elements in $[0,1]$ which do not contain the digits 3 and 9 in their decimal expansion. Is E Lebesgue measurable? If yes find its measure.

b) Show that if $f:\mathbb{R}\to\mathbb{R}$ is measurable then the set $\{x\in\mathbb{R} : \mu(f^{-1}(x))>0\}$ has measure zero.

R - II: Let $f_n:[-1,1]\to\mathbb{R}$ be a sequence of Lebesque measurable functions that converges to f almost everywhere. If $\int_{[-1,1]}|f_n|^4 d\mu \le 1$ for every n then show that $\int_{[-1,1]}|f_n-f|d\mu$ converges to 0.

R - III: Let (X,d) be a compact metric space and let $f:X\to X$ be a continuous function. Show that there exists $A\subseteq X$ a compact subset such that $f(A)=A$.

R - IV: Let $F_k\subset[0,1]$, $k\in\mathbb{N}$ be measurable sets, and there exists $\delta>0$ such that $m(F_k)\ge\delta$ for all k. Assume the sequence $a_k\ge 0$ satisfies

$$\sum_{k=1}^{\infty} a_k \chi_{F_k}(x) < \infty \text{ for a.e. } x\in[0,1].$$

Show that

$$\sum_{k=1}^{\infty} a_k < \infty.$$

Make sure you include all the details in your arguments.

Complex analysis

C - I: Solve at your choice ONE of the following problems:

a) Compute the following integral

$$\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^3 \, dx.$$

b) Let $\mathcal{P}$ be the open region determined by the pentagon with vertices at ω^k where $k = \overline{0,4}$ and $\omega = \cos(2\pi/5) + i\sin(2\pi/5)$. Let $f : \overline{\mathcal{P}} \to \mathbb{C}$ be a continuous function that is analytic on $\mathcal{P}$. Assume that for every $t \in (0,1)$ we have that $\lim_{z \to \frac{2-t+t\omega}{2}} f(z) = \lim_{z \to \frac{2-t\omega^2+t\omega^3}{2}} f(z) = 0$. Find f.

C - II: If we denote by $\mathcal{H} = \{z \in \mathbb{C} : |z - i| > 1\}$ then describe all analytic, bijective maps $f : \mathcal{H} \to \mathcal{H}$.

C - III: Let f be a non-constant, analytic function on the unit disk $\mathbb{D}$. If there exists a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=2}^{\infty} n|a_n| \leq |a_1|$ then show that f is injective.

C - IV: Let f be an analytic function on the open unit disk $\mathbb{D}$. Assume that for every $z \in (-1, 0]$ the power series expansion around z has a vanishing coefficient. Show that f is a a polynomial function.