Ph.D. Qualifying Examination in Analysis

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Instructions. Be sure to put your name on each booklet you use.

This examination has a number of "true-false" questions in it. When a problem is a true-false problem, the operative statement will be preceded by **True-False?**. You are to decide whether it is true or false. If you think it is true, you must provide a proof. If you think it is false, you must provide a counter example or a proof of why it is false. No points will be given for a correct guess that the problem is true or false without any justification. Also, there will be no "Bankruptcy" points given.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus, but still mark the four problems (in each part) that you think are your best work.

Part I

1. Let f be a real-valued function defined on the interval $[0, 1]$. **True-False?** If f is *not* of bounded variation on $[0, 1]$, then there is a point x_0 in $[0, 1]$ such that on any open interval I about x_0, f fails to be of bounded variation on I.

2. A (parametrized) curve C in the plane is given by a pair of real-valued functions f and g defined on an interval $[a, b]$. (So as a point set, $C = \{(f(t), g(t)) \mid t \in [a, b]\}$.) The *length of* C is defined to be
$$\sup\{\sum_{i=1}^{n}[(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2]^{\frac{1}{2}}\},$$
where the sup is taken over all partitions $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. **True-False?** The length of C is finite if and only if f and g are of bounded variation.

3. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open sets in $[0, 1]$. **True-False?** If the interior of $K := \cap_{n=1}^{\infty} U_n$ is empty, then the Lebesgue measure of K is zero.

4. Let f be a non-negative real-valued function defined on the interval $[0, 1]$. **True-False?** f is measurable if and only if there is a (finite or infinite) sequence $\{E_n\}$ of measurable subsets of $[0, 1]$ and a sequence of non-negative constants $\{c_n\}$ such that $f(x) = \sum c_n 1_{E_n}(x)$ for every $x \in [0, 1]$.

5. Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative Lebesgue measurable functions defined on $\mathbb{R}$ that converges almost everywhere (with respect to Lebesgue measure m) to the function f. **True-False?** If $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm = 0$, then $f = 0$ a.e. with respect to m.

Part II

6. Let $P_n(z) := \sum_{k=0}^{n-1}(k + 1)z^k$, $n = 1, 2, \cdots$ and let $0 < r < 1$. **True-False?** There is an n_0 such that for all $n > n_0$, P_n has no zero in the disc $\{|z| < r\}$.

7. Suppose a is an isolated singularity of f and suppose the real part of f, $\Re(f(z))$, satisfies the inequality $\Re(f(z)) \leq -m \ln |z - a|$ for some positive integer m and for z in some disc centered at a. What kind of singularity is a? (Is it removable, a pole, or essential?)

8. Let C be the circle $x^2 + y^2 = 2x$ oriented in the counter clockwise direction. Calculate $\int_C \frac{dz}{z^4+1}$.

9. Let Ω be a region in the plain, $\mathbb{C}$, and let $\mathcal{F}_\Omega = \cup_{n\geq 0}\{f \mid f|\Omega = z^n\}$. Identify all the regions Ω such that $\mathcal{F}_\Omega$ is a normal family.

10. Suppose the series $\sum_{k=0}^{\infty} b_n z^n$ converges in the open unit disc and that $b_n \geq 0$ for all n. Let

$$\mathcal{F} = \left\{ \sum_{k=0}^{\infty} a_n z^n \mid |a_n| \leq b_n \right\}.$$

True-False? $\mathcal{F}$ is a normal family.