Qualifying Exam —Analysis
Summer 2020

Professors Ionut Chifan and Lihe Wang

Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains a section of 5 problems in real analysis and a section of 5 problems in complex analysis. For maximum points you must submit solutions for 7 problems, at least 3 from each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- You are not allowed to use a calculator, cell phone, ipad or any other internet browser device during the exam.

Good luck!

NAME (*PRINT*): ________________________________

Mark in the first column below which problems should be graded!

Your Choice	**Problem**	**Points**	**Your Score**
	R - I	25	
	R - II	25	
	R - III	25	
	R - IV	25	
	R - V	25	
	C - I	25	
	C - II	25	
	C - III	25	
	C - IV	25	
	C - V	25	
	Total	175	

Real Analysis

R - I: Solve at your choice ONE of the following problems:

a) Suppose for all any $x \in (0,1)$ and $\varepsilon > 0$ there exists $0 < r < \varepsilon$, such that $\int_{x-r}^{x+r} f(x)dx \geq 2r$. Show that $f \geq 1$ a.e for $x \in [0,1]$.

b) Is there a closed, uncountable subset of $\mathbb{R}$ containing no rational numbers? Justify your answer!

c) (True-False) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and denote by $A = \{x \in \mathbb{R} : m(f^{-1}(\{x\})) > 0\}$. Then $m(A) = 0$. If you believe is true provide a proof otherwise supply a counterexample.

R - II: (True-False) If f is integrable on $\mathbb{R}$ then $\lim_{x\to\infty} f(x) = 0$. If you believe it is true provide a proof, otherwise supply a counterexample.

R - III: Suppose E is a measurable set such that $m(E \cap (a,b)) \geq \frac{b-a}{2}$ for all $a < b$. Show that E is the whole axis except a measure zero set.

R - IV: Let A be a measurable subset of $[0,2]$ and define $f : \mathbb{R} \to \mathbb{R}$ by letting $f(x) = m((-\infty, x] \cap A)$, for every $x \in \mathbb{R}$; here m is the Lebesgue measure on $\mathbb{R}$.

1.) Show that f is absolutely continuous on $\mathbb{R}$, calculate f' and $\int_0^3 f'(x)dm(x)$, explaining your reasoning.

2.) Show that for every $0 < b < m(A)$ there exists $x_0 \in \mathbb{R}$ such that $b = m((-\infty, x_0] \cap A)$.

Make sure you state correctly all the results you use in the proof.

R - V: Let $1 \leq p < \infty$ and suppose that $f, f_k \in L^p(\mathbb{R})$ are functions satisfying $\lim_{k\to\infty} f_k(x) = f(x)$, for almost every $x \in \mathbb{R}$. Then prove that $\lim_{k\to\infty} \|f_k - f\|_{L^p} = 0$ if and only if $\lim_{k\to\infty} \|f_k\|_{L^p} = \|f\|_{L^p}$.

Complex analysis

C - I: Solve at your choice ONE of the following problems:

a) If $0 < a < 1$ then show that
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}.$$

b) (True-False) Let f be analytic on the open punctured unit disk $D(0,1) \setminus \{0\}$. Can f' have a polar singularity of order one at 0? If you believe it is true provide a proof, if not supply a counterexample. Also make sure you include all the details in your arguments.

c) Assume that $(a_n) = (1, 1, 2, 3, 5, 8, ...)$ is Fibonacci sequence. Consider the power series $f(z) = \sum_n a_n z^n$. Find the radius of convergence for $f(z)$ and determine a singularity point of the circle of convergence in case it is finite.

C - II: Find all entire functions f of finite order such that f has 2020 roots and f' has 2022 roots, counted with their multiplicities. State clearly all the theorems you are using.

C - III: Assume that f is an entire function such that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z) = az^n$ for some integer $n \geq 0$ and some $a \in \mathbb{C}$ with $|a| = 1$.

C - IV: Let $\mathcal{F}$ be the class of all $f \in H(D(0,1))$ such that $Ref > 0$ and $f(0) = 1$. Show $\mathcal{F}$ is a normal family.

C - V: Suppose that $f : D(0,1) \to P$ is a conformal mapping onto a regular pentagonal region P, with center at 0 such that $f(0) = 0$. Compute $f^{(2020)}(0)$.
(Here we denoted by $f^{(n)}$ the n-th derivative of f.)

R - I:Solve at your choice ONE of the following problems:

a) Suppose for all any $x \in (0,1)$ and $\varepsilon > 0$ there exists $0 < r < \varepsilon$, such that $\int_{x-r}^{x+r} f(x)dx \geq 2r$. Show that $f \geq 1$ a.e for $x \in [0,1]$.

b) Is there a closed, uncountable subset of $\mathbb{R}$ containing no rational numbers? Justify your answer!

c) (True-False) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and denote by $A = \{x \in \mathbb{R} : m(f^{-1}(\{x\})) > 0\}$. Then $m(A) = 0$. If you believe is true provide a proof, otherwise supply a counterexample.

Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: R - II:(True-False) If f is integrable on $\mathbb{R}$ then $\lim_{x\to\infty} f(x) = 0$. If you believe it is true provide a proof, otherwise supply a counterexample.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: R - III: Suppose E is a measurable set such that $m(E \cap (a, b)) \geq \frac{b-a}{2}$ for all $a < b$. Show that E is the whole axis except a measure zero set.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: R - IV: Let A be a measurable subset of $[0, 2]$ and define $f : \mathbb{R} \to \mathbb{R}$ by letting $f(x) = m((-\infty, x] \cap A)$, for every $x \in \mathbb{R}$; here m is the Lebesgue measure on $\mathbb{R}$.

1.) Show that f is absolutely continuous on $\mathbb{R}$, calculate f' and $\int_0^3 f'(x)dm(x)$, explaining your reasoning.

2.) Show that for every $0 < b < m(A)$ there exists $x_0 \in \mathbb{R}$ such that $b = m((-\infty, x_0] \cap A)$.

Make sure you state correctly all the theorems you use in the proof.

Solution:

PROBLEM: R - V: Let $1 \leq p < \infty$ and suppose that $f, f_k \in L^p(\mathbb{R})$ are functions satisfying $\lim_{k\to\infty} f_k(x) = f(x)$, for almost every $x \in \mathbb{R}$. Then prove that $\lim_{k\to\infty} \|f_k - f\|_{L^p} = 0$ if and only if $\lim_{k\to\infty} \|f_k\|_{L^p} = \|f\|_{L^p}$. Make sure you state correctly all the theorems you use in the proof.

Solution:

PROBLEM: C - I: Solve at your choice ONE of the following problems:

a) If $0 < a < 1$ then show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}.$$

b) (True-False) Let f be analytic on the open punctured unit disk $D(0,1) \setminus \{0\}$. Can f' have a polar singularity of order one at 0? If you believe it is true provide a proof, if not supply a counterexample. Also make sure you include all the details in your arguments.

c) Assume that $(a_n) = (1, 1, 2, 3, 5, 8, ...)$ is Fibonacci sequence. Consider the power series $f(z) = \sum_n a_n z^n$. Find the radius of convergence for $f(z)$ and determine a singularity point of the circle of convergence in case it is finite.

Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: C - II. Find all entire functions f of finite order such that f has 2020 roots and f' has 2022 roots, counted with their multiplicities. State clearly all the theorems you are using.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: C - III. Assume that f is an entire function such that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z) = az^n$ for some integer $n \geq 0$ and some $a \in \mathbb{C}$ with $|a| = 1$.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: C - IV. Let $\mathcal{F}$ be the class of all $f \in H(D(0,1))$ such that $Ref > 0$ and $f(0) = 1$. Show $\mathcal{F}$ is a normal family.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: C - V. Suppose that $f : D(0,1) \to P$ is a conformal mapping onto a regular pentagonal region P, with center at 0 such that $f(0) = 0$. Compute $f^{(2020)}(0)$.
(Here we denoted by $f^{(n)}$ the n-th derivative of f.)
Make sure you state correctly all the results you use in the proof.

Solution: