Factoring Polynomials and Solving Quadratic Equations

Math Tutorial Lab Special Topic*

Factoring

Factoring Binomials

Remember that a **binomial** is just a polynomial with two terms. Some examples include $2x+3$ and $6x^2+7x$. If the terms in a binomial expression share a common factor, we can rewrite the binomial as the product of the common factor and the rest of the expression. This process is called factoring.

Example Factor the expression completely: $7x - 14$.

Example Factor the expression completely: $8x^2 + 4x$.

Factoring Trinomials with 1 as the Leading Coefficient

Much like a binomial, a **trinomial** is a polynomial with three terms. Most of the examples we'll give here will be quadratic – that is, they will have a squared term. Some examples include x^2+5x+4 and $2x^2+3x-2$. When factoring these expressions, our goal will be to write the trinomial as the product of two binomials.

When thinking about how to factor a quadratic, we want to keep the following in mind. Consider $(x+a)(x+b)$, the product of two binomials. If we were to multiply this product, we'd have $(x+a)(x+b) = x^2+bx+ax+ab = x^2 + (a + b)x + ab$. So, when trying to factor, we want to find two numbers, an a and a b, such that when we add a and b, we get the coefficient on the x term. Similarly, when we multiply a and b, we should get the constant term at the end of the quadratic.

Example Factor $x^2 - 8x - 9$.

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Example Factor $x^2 - 12x + 27$.

Difference of Squares

One special case when factoring is called the difference of squares. Let's look at the general example $(c + d)(c - d)$. If we multiply this, we have $c^2 - cd + cd - d^2 = c^2 - d^2$. Here, we're taking the difference of two squared terms, and hence the name difference of squares. Thus, whenever you see an expression that looks like $c^2 - d^2$, you can factor it as $(c + d)(c - d)$.

Example Factor $x^2 - 16$.

Example Factor $49x^2 - 9$.

Factoring Trinomials with a Leading Coefficient Other than 1

You may have noticed that the coefficient on the x^2 term was just 1 in many of the above examples. Can we still factor if we have a coefficient other than 1 on the x^2 term? We can. In this case, we have a couple options. First, if all the coefficients have a common factor, you may try to factor out the leading coefficient, leaving behind a quadratic with a leading 1 coefficient. If you can't factor out the leading coefficient in this way, you can still factor using a similar process to that above. This time, however, we will look at the factors of the leading coefficient as well as the factors of the constant term. In addition, looking at the signs on the x term as well as the sign on the constant term will provide more information on how we should factor the quadratic. It may seem complicated at first, but with practice, these steps will become much clearer.

Example Factor $2x^2 + 8x + 8$.

Example Factor $125x^2 - 20$.

Example Factor $4x^2 + 25x - 21$.

Example Factor $6x^2 + 7x + 1$.

Factoring by Grouping

This factoring technique is useful for factoring polynomials with order higher than 2 (the largest power on x is larger than 2). You can also use this method if you have an expression containing more than one variable. When using this method, we want to "group" terms together that have a common factor. Then, after factoring the common factor out of these groups, we will be left with an expression that has a binomial as a common factor. If we then factor once more, we'll have our fully factored expression.

Example Factor $x^3 + 2x^2 + 8x + 16$.

Example Factor $xy - 5y - 2x + 10$.

Sums and Differences of Cubes

Many of our factoring techniques have focused on quadratic expressions. We'll now give a brief overview of some common factorizations of cubic functions. If we have a cubic expression that looks like $a^3 + b^3$ (sum of cubes), we can factor as $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. Similarly, if we have an expression that looks like $a^3 - b^3$ (difference of cubes), we can factor as $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Example Factor $27x^6 + 125$.

Example Factor $40c^3 - 5d^3$.

It may also be useful to recognize what a binomial cubed looks like when expanded. We explore this in the following example.

Example Expand $(x + 2)^3$.

Solving Quadratic Equations

Solving by Taking a Square Root

If the equation we're trying to solve has the variable squared, and only constants everywhere else, then we can solve the equation by isolating the squared variable on one side of the equation and taking a square root. Remember that taking a square root produces both a positive square root and a negative square root. For example, $\sqrt{4} = \pm 2$. Therefore, whenever solving a quadratic equation, you will typically have two solutions.

Example Solve $2x^2 + 3 = 75$.

Example Solve $(4x - 1)^2 - 8 = 0$.

Solving by Factoring

Many times, the equations we want to solve will not be as nice as the ones above. But, we can make use of the factoring techniques we learned in order to solve these equations as well. If our equation has a quadratic trinomial, we can move all variables and constants to one side of the equation (we have to have one side of the equation equal to zero to use this method). Then, we factor the quadratic, and set each factor equal to zero to solve for our two solutions.

Example Solve $x^2 - 2x - 35 = 0$.

Example Solve $3x^2 - 2x = 8$.

Solving by the Quadratic Formula

One last method for solving quadratic equations is the quadratic formula. This formula can be used on any quadratic with the form $ax^2 + bx + c = 0$. Using the coefficients in the quadratic, the formula (derived from the process of completing the square) tells you the roots or zeros of the quadratic. The formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The formula can be used for any of the examples we've discussed previously, but is especially useful for quadratics with imaginary roots, or even real roots that don't simplify very nicely.

Example Solve $-7x^2 + 2x + 9 = 0$.

Example Solve $-x^2 + 8x = 1$.

References

Many ideas and problems inspired by `www.khanacademy.org`.