Graphing Review

Math Tutorial Lab Special Topic*

Common Functions and Their Graphs

Linear Functions

A function f defined by a linear equation of the form $y = f(x) = mx + b$, where m and b are constants, is called a **linear function**. Linear functions have graphs that are straight lines, and any nonvertical straight line is the graph of a linear function.

We can compute the **slope**, m, of a line passing through (x_1, y_1) and (x_2, y_2), when $x_1 \neq x_2$, with the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. If $y_1 = y_2$ $(m = 0)$, the line is horizontal. Horizontal lines have the form $y = a$ for some constant a. If $x_1 = x_2$ (m is undefined), the line is vertical. Vertical lines have the form $x = c$ for some constant c. **Warning:** A vertical line is not a function!

In our definition above, the equation $y = mx + b$ is often called the **slope-intercept** form of a line, since m is the slope and b is the y-intercept of the line. If a line passes through the point (x_1, y_1), we compute b by $b = y_1 - mx_1$.

In addition, a line with slope m that passes through the point (x_1, y_1) has the **point-slope** equation $y - y_1 = m(x - x_1)$. Lastly, linear equations are often written in their general form, $Ax + By + C = 0$, for constants A, B, and C.

Example A line passes through the points $(2, 3)$ and $(-4, 0)$.

(a) Find a point-slope equation of this line.

(b) Find the slope-intercept equation of this line.

In the above example, notice that reversing the order of the points when computing the slope will still give the same slope. Similarly, we could have used the other point to find the point-slope equation. The point-slope equation would look slightly different, but would still reduce to the same slope-intercept equation.

*Created by Maria Gommel, June 2014.

Example Sketch the graph of the linear function described by $f(x) = 2 - \frac{2}{3}x$.

Quadratic Functions

A function f defined by a quadratic equation of the form $y = f(x) = ax^2 + bx + c$, where $a \neq 0$, is a **quadratic function**, and its graph is a **parabola**. The most basic parabola is defined by $y = f(x) = x^2$, whose graph is shown in Figure 1.

The equation $y = f(x) = ax^2 + bx + c$ is often called the **standard form** of the parabola. Quadratic equations can also be written in **vertex form**: $y = f(x) = a(x - h)^2 + k$, where (h, k) is the vertex of the parabola. The constant a stretches the graph vertically if $|a| > 1$ and compresses the graph vertically if $|a| < 1$. Oftentimes, it is easier to graph a quadratic function in vertex form. Converting from standard form to vertex form usually requires completing the square, as will be seen in the example problems.

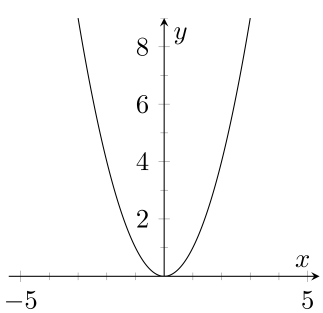


Figure 1

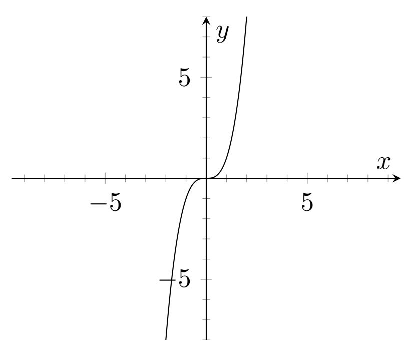


Figure 2

Example Sketch the graph of the quadratic equation $y = (x - 1)^2 + 3$.

Cubic Functions

A function f defined by a cubic equation of the form $y = f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$, is a **cubic function**. The most basic cubic function is defined by $y = f(x) = x^3$, whose graph is shown in Figure 2.

Square Root Function

The graph of the **square root function**, $f(x) = \sqrt{x}$, is shown in Figure 3. Square roots are defined only for nonnegative numbers, so $f(x) = \sqrt{x}$ is defined only for $x \geq 0$ and the domain of the square root function is the set of all nonnegative real numbers.

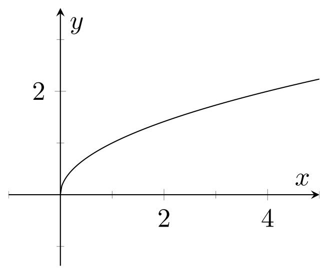


Figure 3

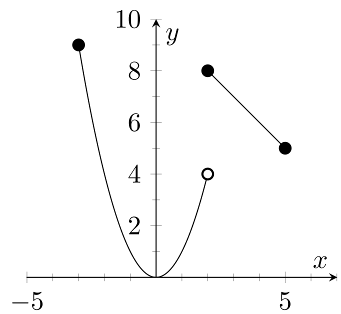


Figure 4

Piecewise-Defined Functions

A function that is defined by differing expressions on various portions of its domain is called a **piecewise-defined** function.

An example of a piecewise-defined function is given by

$$f(x) = \begin{cases} x^2, & \text{if} \quad -3 \leq x < 2 \\ 10 - x, & \text{if} \quad 2 \leq x \leq 5, \end{cases}$$

whose graph is seen in Figure 4.

Graph Transformations

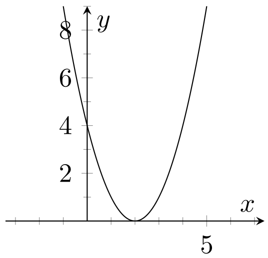
Many times, we may want to sketch a graph by hand. This is most often done by identifying a "parent" function, and then performing various graph transformations to obtain the graph we want. The various types of graph transformations are listed below, followed by a few example problems.

Horizontal Shifts

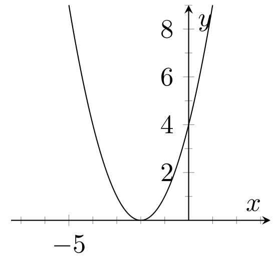
Suppose that $c > 0$.

- The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted right c units.
- The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted left c units.

Example Compare the following graphs with the graph of x^2.



$f(x) = (x - 2)^2$



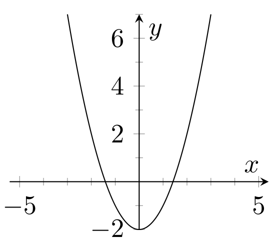
$f(x) = (x + 2)^2$

Vertical Shifts

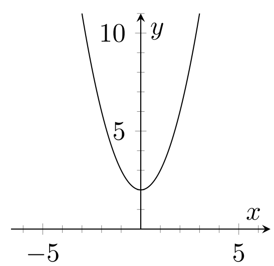
Suppose $c > 0$.

- The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted up c units.
- The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted down c units.

Example Compare the following graphs with the graph of x^2.



$f(x) = x^2 - 2$

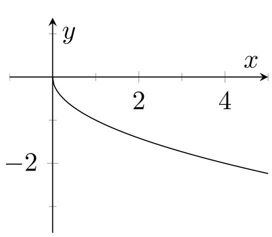


$f(x) = x^2 + 2$

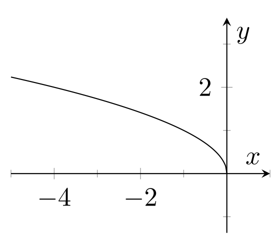
Reflecting over the axes

- The graph of $y = -f(x)$ flips the graph of $y = f(x)$ over the x axis.
- The graph of $y = f(-x)$ flips the graph of $y = f(x)$ over the y axis.

Example Compare the following graphs with the graph of $\sqrt{x}$.



$f(x) = -\sqrt{x}$

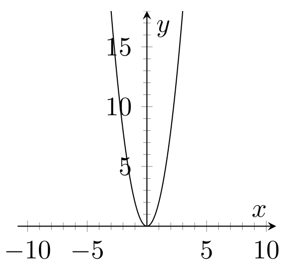


$f(x) = \sqrt{-x}$

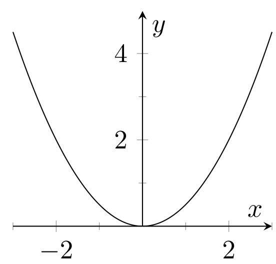
Stretching or Compressing the graph

- The graph of $y = cf(x)$ stretches the graph of $y = f(x)$ vertically when $c > 1$, and compresses the graph vertically when $0 < c < 1$.
- The graph of $y = f(cx)$ compresses the graph of $y = f(x)$ horizontally when $c > 1$ and stretches the graph horizontally when $0 < c < 1$.

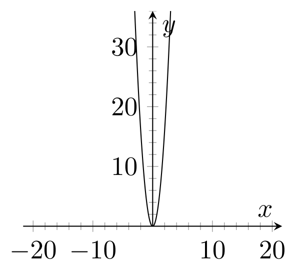
Example Compare the following graphs with the graph of x^2.



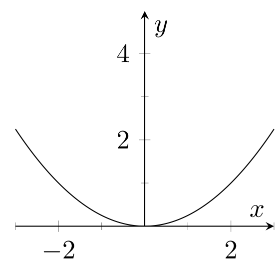
$f(x) = 2x^2$



$f(x) = \frac{1}{2}x^2$



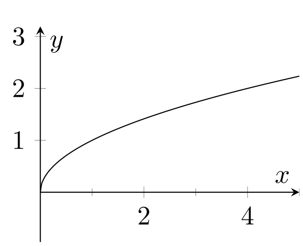
$f(x) = (2x)^2$



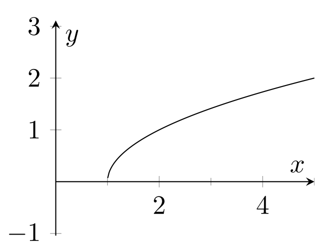
$f(x) = (\frac{1}{2}x)^2$

Example We will give an example of how to graph a rather complicated function using all the transformations above. Consider $f(x) = -2\sqrt{x-1} + 3$.

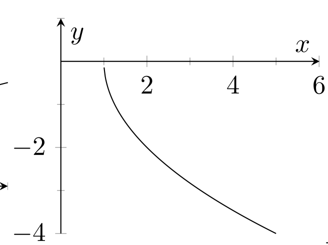
- Step 1: We first identify the "parent" function. Since we have a square root in our function, $\sqrt{x}$ is our parent function. We graph $\sqrt{x}$ first, and then transform the graph according to the rules above and the equation of $f(x)$.
- Step 2: We follow the order of operations. Think of the $x - 1$ under the square root as an expression in parentheses: we'll perform this transformation first. The $x - 1$ signals a horizontal shift to the right by one unit. We shift the graph of $\sqrt{x}$ to the right by one.
- Step 3: Now, again following the order of operations, we'll consider the -2 in front of the square root. Remember that the negative sign in this position flips the graph over the x axis, and the 2 stretches the graph vertically.
- Step 4: Lastly, we look at the $+3$ at the end of the equation. This is a vertical shift of 3 units upwards. These steps are shown in the graphs below.



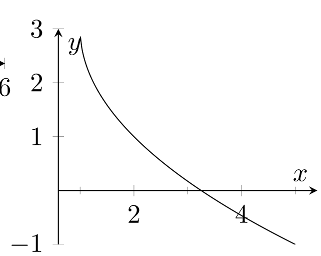
Step 1: $f(x) = \sqrt{x}$



Step 2: $f(x) = \sqrt{x-1}$



Step 3: $f(x) = -2\sqrt{x-1}$



Step 4: $f(x) = -2\sqrt{x-1}+3$

Example Problems

Example Sketch the graph of $f(x) = x^2 - 2x + 4$. (Hint: compare to the example in the Quadratic Functions section).

Example Sketch the graph of $f(x) = 2x^2 + 12x + 17$.

Example Sketch the graph of $f(x) = -x^3 - 1$.

Example Sketch the graph of $f(x) = \sqrt{-x-2}$.

Example Sketch the graph of

$$f(x) = \begin{cases} 2x^2, & \text{if} \quad -4 \le x < 0 \\ \sqrt{x}, & \text{if} \quad 0 \le x \le 4. \end{cases}$$

References

Most definitions, examples, and other text was taken from the fifth edition of *PreCalculus* by J. Douglas Faires and James DeFranza.