Qualifying Exam: PDE, Fall, 2017

Choose any three out of the five problems. Please indicate your choice. Show all your work.

1. Find the solution to the initial value problem

$$u_t + (\frac{u^2}{2})_x = 0, \quad x \in R, \ t > 0,$$

$$u(x,0) = \begin{cases} 2 & x \le 0 \\ 2 - x & 0 < x \le 2 \\ 1 & 2 < x \end{cases} .$$

2. Show that if the C^1 initial data $f(x)$ has $f'(x_0) < 0$ for some x_0 and that $F''(u) \geq 1$ for all u, then the C^1 solution of

$$u_t + F(u)_x = 0, \quad u(x,0) = f(x)$$

must break down at some time $t > 0$.

3. Let $u(x,t)$ and $v(x,t)$ be solutions of the equation

$$u_t - ku_{xx} = q(x,t), \quad x \in R, \quad 0 < t \leq T$$

satisfy

$$u(x,0) = f(x),\ v(x,0) = g(x),\ x \in R$$

respectively, where $k > 0, T > 0$, and $f(x), g(x), q(x,t)$ are continuous and bounded functions. Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in R$, $0 \leq t \leq T$, and that

$$f(x) \leq g(x),\ x \in R.$$

Show that $u(x,t) \leq v(x,t)$ for $x \in R$, $0 \leq t \leq T$.

4. Solve the initial-boundary-value problem

$u_t = u_{xx}, \;\; 0 < x < 1, \; t > 0,$

$u(0,t) = 0, \;\; u(1,t) = 1, \;\; t > 0,$

$u(x,0) = x^2, \;\; 0 < x < 1.$

Also find a steady-state solution $U(x)$ of the above problem.

5. Consider the damped wave equation problem

$u_{tt} + du_t - c^2 u_{xx} = 0, \;\; x \in R, \; t > 0,$

$u(x,0) = f(x), \;\; u_t(x,0) = g(x), \;\; x \in R$

where $c > 0$, $d > 0$ and f, g are smooth functions with compact support.

Define energy as $e(t) = \dfrac{1}{2}\displaystyle\int_R (u_t^2(x,t) + c^2 u_x^2(x,t))dx$.

Show that the energy is nonincreasing as t increases.