Qualifying Exam: PDE, Spring, 2018

Choose any three out of the five problems. Please indicate your choice. Show all your work.

1. (i) Solve the initial value problem

$$u_t - \frac{1}{2x}u_x = -u, \quad x > 0, \; t > 0,$$
$$u(x,0) = \frac{1}{2+x^2}, \quad x \geq 0.$$

Over what region in the first quarter of the x-t plane does the solution exist? Draw the characteristics on the x-t plane where the solution exists.

(ii) Write an upwind scheme for the above problem. What is the CFL condition for the scheme?

2. Find a weak solution for the nonlinear conservation law with the following Riemann initial data such that the discontinuous solutions satisfy the entropy condition

$$u_t + (u(1-u))_x = 0, \quad x \in \mathbb{R}, \ t > 0,$$

(i) with initial data

$$u(x,0) = \begin{cases} 3 & x < 0 \\ 2 & x \geq 0. \end{cases}$$

(ii) with initial data

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 3 & x \geq 0. \end{cases}$$

3. Let $u(x,t)$ and $v(x,t)$ be solutions of the equation

$$u_t - u_{xx} = 2, \quad x \in \mathbb{R}, \quad 0 < t \leq T$$

satisfying

$$u(x,0) = f(x),\ v(x,0) = g(x),\ x \in \mathbb{R}$$

respectively, where $T > 0$, and $f(x), g(x)$ are continuous and bounded functions. Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$ and that

$$f(x) \leq g(x),\ x \in \mathbb{R}.$$

Show that $u(x,t) \leq v(x,t)$ for $x \in \mathbb{R}$, $0 \leq t \leq T$.

4. Solve the initial-boundary-value problem

$u_t = u_{xx}, \;\; 0 < x < 1, \; t > 0,$

$u(0, t) = 1, \;\; u(1, t) = 3, \;\; t > 0,$

$u(x, 0) = x, \;\; 0 < x < 1.$

and also find the steady-state solution $U(x)$ of the above problem.

5. Solve the initial value problem of the wave equation

$u_{tt} - u_{xx} = 0, \;\; x \in \mathbb{R}, \; t > 0,$

$u(x,0) = -e^{-x^2}, \;\; u_t(x,0) = 6xe^{-x^2}, \;\; x \in \mathbb{R}.$