Qualifying Exam: PDE, Fall, 2019

Choose any **Four** out of the five problems. Please indicate your choice.
Show all your work.

1. Find a weak solution for the nonlinear conservation law with the following Riemann initial data such that the discontinuous solutions satisfy the entropy condition

$$u_t + (u(2-u))_x = 0, \quad x \in \mathbb{R}, \ t > 0,$$

(i) with initial data

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0; \end{cases}$$

and

(ii) with initial data

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

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2. (i) Solve the initial value problem

$u_t - 3x^2 u_x = -u, \quad x \in \mathbb{R},\ t > 0,$

$u(x,0) = e^{-2x^2}, \quad x \in \mathbb{R}.$

(ii) Draw the characteristics and find the region in the x-t plane where the solution exists.

(iii) Write an upwind scheme for the above problem.

3. Let both $u(x,t)$ and $v(x,t)$ be solutions of the equation

$$u_t - ku_{xx} = q(x,t), \quad x \in \mathbb{R}, \quad 0 < t \leq T$$

satisfying

$$u(x,0) = f(x) \text{ and } v(x,0) = g(x), \ x \in \mathbb{R}$$

respectively, where $k > 0$, $T > 0$, $f(x), g(x)$ and $q(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$.

Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$, and that

$$f(x) \leq g(x), \ x \in \mathbb{R}.$$

Show that $u(x,t) \leq v(x,t)$ for $x \in \mathbb{R}$, $0 \leq t \leq T$.

4. (i) Solve the initial-boundary-value problem

$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0,$

$u(0,t) = 1, \quad u(1,t) = 3, \quad t > 0,$

$u(x,0) = x^2 + x + 1, \quad 0 \le x \le 1.$

(ii) What is the limit of the solution as $t \to +\infty$?

5. Solve the following initial-boundary-value problem

$u_{tt} - u_{xx} = 0, \;\; x > 0, \; t > 0,$

$u(x, 0) = f(x), \;\; u_t(x, 0) = g(x), \;\; x \geq 0,$

$u_x(0, t) = 1, \;\; t > 0$

where f and g are smooth functions satisfying $f'(0) = 1$ and $g'(0) = 0$.