Qualifying Exam: PDE, Spring, 2019

Choose any **Four** out of the five problems. Please indicate your choice. Show all your work.

1. Show that if the C^1 initial data $f(x)$ has $f'(x_0) < 0$ for some x_0, then the C^1 solution of

$$u_t + (u^2)_x = 0, \quad x \in \mathbb{R}, t > 0, \quad u(x,0) = f(x)$$

must break down at some time $t > 0$.

2. (i) Solve the initial value problem

$u_t + x^2 u_x = -u, \quad x \in \mathbb{R},\ t > 0,$

$u(x, 0) = x^2, \quad x \in \mathbb{R}.$

(ii) Over which region in the x-t plane does the solution exist?

(iii) Write an upwind scheme for the above problem.

3. Solve the following initial boundary value problem

$u_t - u_{xx} = 0, \quad x > 0, \ t > 0,$

$u(x, 0) = f(x), \quad x \geq 0,$

$u(0, t) = 1, \quad t \geq 0$

where $f \in C^2[0, +\infty)$ is bounded and $f(0) = 1$.

4. (i) Solve the initial-boundary-value problem

$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0,$

$u(0,t) = 0, \quad u(1,t) = 3, \quad t > 0,$

$u(x,0) = x^2 + 2x, \quad 0 \leq x \leq 1.$

(ii) What is the limit of the solution as $t \to +\infty$?

5. Consider the damped wave equation problem

$u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in R, \ t > 0,$

$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in R$

where $c > 0$, $d > 0$ and f, g are smooth functions with compact support. Define energy as $e(t) = \frac{1}{2} \int_R (u_t^2(x, t) + c^2 u_x^2(x, t)) dx.$

Show that the energy is nonincreasing as t increases.