**Qualifying Exam: PDE, Fall, 2020**

Choose any **Four** out of the five problems. Please indicate your choice. Show all your work.

**1.** Find a weak solution for the nonlinear conservation law with the following Riemann initial data such that the discontinuous solutions satisfy the entropy condition

$$u_t + (u(1+u))_x = 0, \quad x \in \mathbb{R},\ t > 0,$$

(i) with initial data

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0; \end{cases}$$

and

(ii) with initial data

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

**2**. (i) Solve the initial value problem

$$u_t + x^2 u_x = -u, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = e^{-x^2}, \quad x \in \mathbb{R}.$$

(ii) Draw the characteristics and find the region in the $x$-$t$ plane where the solution exists.

(iii) Write an upwind scheme for the above problem.

**3.** Let $u(x,t)$ and $v(x,t)$ be solutions of the heat equation

$$u_t - ku_{xx} = 1, \quad x \in \mathbb{R}, \quad 0 < t \leq T$$

satisfying

$$u(x,0) = f(x) \text{ and } v(x,0) = g(x),\ x \in \mathbb{R}$$

respectively, where $k > 0$, $T > 0$, $f(x)$ and $g(x)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$.

Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$, and that

$$f(x) \leq g(x),\ x \in \mathbb{R}.$$

Show that $u(x,t) \leq v(x,t)$ for $x \in \mathbb{R}$, $0 \leq t \leq T$.

**4.** (i) Solve the initial boundary value problem

$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0,$

$u(0,t) = 2, \quad u(1,t) = 3, \quad t > 0,$

$u(x,0) = x^2 + 2, \quad 0 \leq x \leq 1.$

(ii) What is the limit of the solution as $t \to +\infty$?

**5.** (i) Solve the initial boundary value problem

$u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \quad t \in \mathbb{R},$

$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \geq 0,$

$u_x(0,t) = 0, \quad t \in \mathbb{R}$

where $c > 0$, $f \in C^2$, $g \in C^1$, $f'(0) = 0$ and $g'(0) = 0$.

(ii) Assuming further that $f, g$ are of compact support, i.e., $f(x) = g(x) = 0$ for $|x| > a$, for some $a > 0$, show that the energy

$$e(t) = \frac{1}{2} \int_0^{+\infty} (u_t^2(x,t) + c^2 u_x^2(x,t)) dx$$

is a conserved quantity as $t$ varies.