Ph.D. Qualifying Exam in Topology

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Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A

1. Prove that for every group $G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$ there is a 2-dimensional cell-complex X_G with $\pi_1(X_G) \simeq G$.

2. (a) Let α be an oriented closed loop in a topological space X. Let $x_0 \in X$ and $p, q \in \alpha$ be points. (They are possibly the same points.) Let γ (resp. δ) be an oriented path in X that starts at x_0 and ends at p (resp. q).

   Show that homotopy classes $[\gamma * \alpha * \overline{\gamma}]$ and $[\delta * \alpha * \overline{\delta}]$ in $\pi_1(X, x_0)$ are conjugate to each other.

   (b) Let x_0 and x_0' be points in X. Show that $\pi_1(X, x_0)$ and $\pi_1(X, x_0')$ are group isomorphic.

3. Let D be the unit disk in $\mathbb{R}^2$. Show that every continuous map $h : D \to D$ has a fixed point.

4. Let $i = 1, 2$. Let $h_i : D^2 \times S^1 \to A_i$ be a homeomorphism. The boundary of A_i is a torus $\partial A_i = \partial D^2 \times S^1 = S^1 \times S^1$. We define two curves on ∂A_i, a meridian m_i and a longitude l_i, as follows: Fix points $x \in \partial D^2$ and $y \in S^1$. The meridian is defined by $m_i = h_i(\partial D^2 \times \{y\})$ and the longitude is defined by $l_i = h_i(\{x\} \times S^1)$.

   Let (p, q) be co-prime positive integers. Let $\phi : \partial A_1 \to \partial A_2$ be a homeomorphism that takes the meridian m_1 to a simple closed curve whose homotopy type is equal to $p[m_2] + q[l_2]$. Construct a space X gluing the solid tori A_1 and A_2 along their boundary using the map ϕ.

   Using the van Kampen Theorem compute the fundamental group of the space X.

5. Let $X = S^1 \vee S^1$.

   (a) Find a 3:1 (non-trivial) normal covering of X and find the corresponding normal subgroup of $\pi_1(X)$.

   (b) Find a 3:1 non-normal connected covering of X.

   (c) Describe the universal covering of X.

6. (a) Describe a cell-complex structure on the n-dimensional real projective space $\mathbb{R}P^n$ where $n \geq 2$.

(b) Find a non-trivial covering of $\mathbb{R}P^n$ and compute its deck transformation group.

Part B

1. Let S be the unit sphere in $\mathbb{R}^3$. Find a C^∞ atlas on S that consists of two charts.

2. State the Regular Level Set Theorem, then show that the unit sphere S is a 2-dimensional manifold.

3. Let U be an open subset of $\mathbb{R}^n$. Show that the exterior derivative $d : \Omega^*(U) \to \Omega^*(U)$ satisfies the *Cocycle Condition*; that is, $d^2 = 0$.

4. (a) Show that the map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\phi(x, y, z) = (2y, -x, -xy + z)$$

is a diffeomorphism.

(b) Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be a vector field on $\mathbb{R}^3$. Compute $\phi_*(X)$ at $p = (x, y, z)$.

(c) Let $\alpha = dz - ydx$ be a 1-form on $\mathbb{R}^3$. Compute the pull back $\phi^*(\alpha)$ at $p = (x, y, z)$.

5. Let ω be a 2-form on the unit sphere S in $\mathbb{R}^3$ defined by:

$$\omega = \begin{cases} \frac{dy \wedge dz}{x} & \text{for } x \neq 0, \\ \frac{dz \wedge dx}{y} & \text{for } y \neq 0, \\ \frac{dx \wedge dy}{z} & \text{for } z \neq 0. \end{cases}$$

Show that this ω is well-defined. (In other words, if $x, y, z \neq 0$ then the three expressions give the same 2-form.) Then compute the integral $\int_S \omega$.

6. Let $O(n)$ be the orthogonal group, the group of linear transformations of $\mathbb{R}^n$ that preserve distance. In other words, the group of $n \times n$ matrices such that $AA^t = I$ where A^t is the transpose of A and I is the identity matrix. Show that $O(n)$ is a manifold of dimension $n(n-1)/2$.