Ph.D. Qualifying Exam in Topology

Ben Cooper, Mohammad Farajzadeh

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Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A - General Topology

1. Suppose $f\colon A \to Y$ is a continuous function from $A \subset X$, with subspace topology, to a Hausdorff topological space Y. Show that a continuous extension $\overline{f}\colon \overline{A} \to Y$ is unique (if it exists). Provide a non-Hausdorff example where more than one extensions exist.

2. Show that every compact metrizable space is 2nd countable (i.e., has a countable basis).

3. Given topological spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of continuous maps from X to Y.

   - Let $I = [0, 1]$. Show that $[X, I]$ has a single element.
   - Under what necessary and sufficient condition on Y the set $[I, Y]$ has a single element.

4. Give a topological proof that every square matrix with positive real entries has a positive eigenvalue.

Part A - Algebraic Topology

1. Give cell-decompositions of $\mathbb{R}P^2$ and $K =$ the Klein bottle, and compute their fundamental groups.

2. Find the universal covering space of $\mathbb{R}P^n$ ($n > 1$), then compute the deck transformation group of the universal covering.

3. Let $f : D^2 \to D^2$ be a homeomorphism. Prove that f has a fixed point.

4. Let A and B be homeomorphic to the solid torus $D^2 \times S^1$. Glue A and B along their boundary surfaces $\partial A \simeq \partial B \simeq S^1 \times S^1$ using a homeomorphism $\phi : \partial A \to \partial B$ so that the slope ∞ curve on ∂A and the slope $5/2$ curve on ∂B are identified. See the figure. Compute the fundamental group of the resulting 3-dimensional space.

Part B

1. (a) Define a covering space $\pi : E \to M$

   (b) Prove that if M is a smooth manifold then E is a smooth manifold.

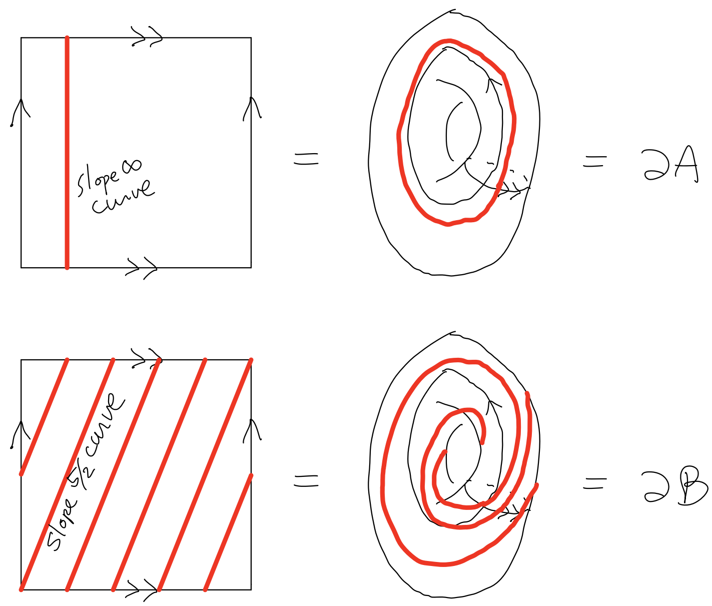

Figure 1: Curves on ∂A and ∂B

2. (a) Let M be a smooth manifold. Recall the total space T^*M of the cotangent bundle $\pi : T^*M \to M$ has local coordinates (p, φ) where $p \in M$ and $\varphi : T_pM \to \mathbb{R}$. Let $\theta \in \Omega^1(T^*M)$ be the 1-form defined by

$$\theta(X)_{(p,\varphi)} := \varphi(\pi_*(X))$$

where $\pi_* : TT^*M \to TM$ is the Jacobian of the map π and $X \in \mathcal{X}(T^*M)$ is any vector field on T^*M. Find an expression for θ in local coordinates when $M = \mathbb{R}^n$.

3. Prove that if M is a smooth manifold and $\pi_1(M)$ has no non-trivial index 2 subgroup then M is orientable.

4. If $\omega := xdxdy + ydydz + zdzdx \in \Omega^2(\mathbb{R}^3)$ and

$$X := yz\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}$$

then compute the Lie derivative $\mathcal{L}_X\omega$ of ω with respect to the vector field X.

5. (a) Prove that the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$$

is a smooth manifold.

(b) If $x \in S^n$ then prove that there is an identification $x^\perp \cong T_xS^n$ which extends to an isomorphism of vector bundles.

6. (a) Define smooth manifold with boundary.

(b) State Stokes' Theorem

(c) Suppose that $S \subset M$ is a regular submanifold of dimension n, let $i : S \to M$ be the inclusion. Prove that $\eta \in \Omega^{n-1}(M)$ is a differential form. Prove that

$$\int_S i^*(d\eta) = 0$$