Ph.D. Qualifying Exam in Topology

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Instructions.

- Do eight problems: four from part A and four from part B.
- This is a closed book examination: you should have no books, technology or paper of your own.
- Please do your work on the paper provided according to the format outlined below.
  - On each page of your solutions
    - Write your name
    - Write the page number
    - Indicate which problem is being addressed
  - When you start a new problem, start a new page
  - Only write on one side of the paper
  - Make a cover page and indicate which eight problems you want graded.
- Always justify your answers unless explicitly instructed otherwise.
- You may use theorems if the problem is not a step in proving that theorem. You must to state any theorems that you use clearly and carefully.

Part A - Algebraic Topology

1. (a) Find an action of the group $\mathbb{Z}^2$ on the space $\mathbb{R}^2$ and prove that it satisfies the following condition:

   For every point $\vec{\mathbf{x}} = (x, y) \in \mathbb{R}^2$ there exists an open set $U \subset \mathbb{R}^2$ containing $\vec{\mathbf{x}}$ such that the images $\{g(\vec{\mathbf{x}}) \mid g \in \mathbb{Z}^2\}$ are pairwise disjoint.

   (b) Construct a normal covering using the above group action. (Need to show your covering is normal.)

2. Consider 2-dimensional surfaces.

   (a) Find the fundamental group of a sphere.

   (b) Find the fundamental group of a torus.

   (c) Find the fundamental group of a 4-punctured sphere (a sphere minus four distinct points).

   (d) Find the fundamental group of a 4-punctured torus.

   (e) Find the fundamental group of a genus 2 surface.

3. (a) Find a 2:1 covering map from a 4-punctured torus to a 4-punctured sphere.

   (b) Find the deck transformation group of the above covering map.

4. (a) Find the universal covering space of a torus.

   (b) Find the universal covering space of a 1-punctured torus.

5. Answer the questions using van Kampen's theorem.

   (a) Let X and Y be genus 2 solid handle bodies. Let α and β be curves on ∂X. Let $f : \partial X \to \partial Y$ be a homeomorphism of a genus 2 surface such that the images $f(\alpha)$ and $f(\beta)$ of the curves α and β are as depicted in the figure. Glue X and Y along their boundary surfaces using the map f. Find the fundamental group of the resulting space $(X \cup Y)/f$.

   (b) Let $g : \partial X \to \partial Y$ be another homeomorphism such that the images $g(\alpha)$ and $g(\beta)$ are as depicted in the figure. Find the fundamental group of the resulting space $(X \cup Y)/g$.

6. Let Y be a Y-shape graph (three edges meeting at a one vertex). Take a product $Y \times [0,1]$ and identifying the two ends ($Y \times \{0\}$ and $Y \times \{1\}$) via a one-third twist as shown in the figure. Call the CW complex Z. Show that Z has a circle boundary. Then attach a disk to Z along the circle boundary. Compute the fundamental group of the resulting CW complex.

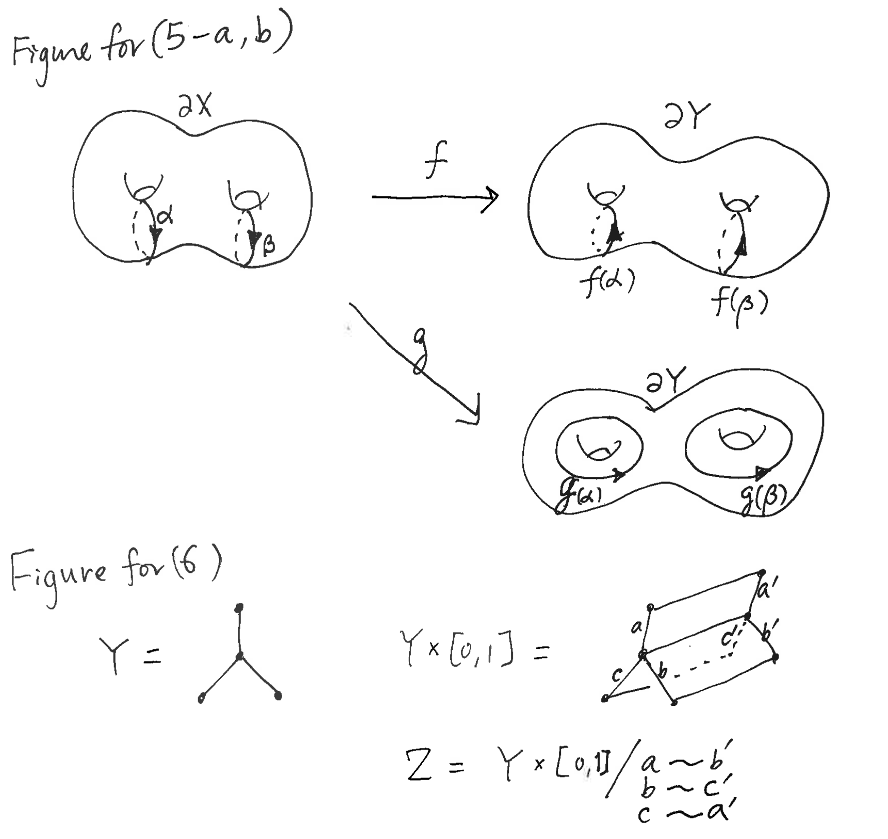

Figure 1: Figures for problems 5 and 6.

Part B

1. (a) Define a smooth n-dimensional vector bundle $\pi : E \to M$. Define a vector bundle trivial.

   (b) Prove that a vector bundle $E \to M$ of dimension n determines a 1-dimensional vector bundle $\Lambda^n(E) \to M$.

   (c) Using any definition of orientable studied in class or the textbook, prove or disprove the statement that M is orientable if and only if $\Lambda^n(T^*M)$ is the trivial bundle.

2. (a) Define a smooth manifold.

   (b) If the circle $S^1 \subset \mathbb{R}^2$ is defined by the subspace topology

   $$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

   then, using only the definition of smooth manifold from problem 2. (a), prove that the n-torus $(S^1)^{\times n}$ is a smooth manifold for all $n \geq 1$.

3. (a) If the sphere $S^2(r) \subset \mathbb{R}^3$ of radius $r > 0$ is defined by the subspace topology

   $$S^2(r) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$$

   then prove that the sphere $S^2(r)$ is a smooth manifold.

   (b) Compute the integral

   $$\int_{S^2} (xdydz - zdzdy + ydxdy).$$

4. Let V be a real vector space of dimension n.

   (a) Define the vector spaces $\Lambda^k(V^*)$ for $k = 0, \ldots, n$.

   (b) Recall that a bilinear form $\omega : V \times V \to \mathbb{R}$ is *non-degenerate* if for each $x \in V$, $\omega(x, y) = 0$ for all $y \in V$ implies $x = 0$.

   Prove that $\omega \in \Lambda^2(V^*)$ is non-degenerate if and only if the map $\varphi : V \to V^*$ given by $\varphi(x) = \iota_x \omega$ is an isomorphism.

(c) Recall that a bilinear form is *degenerate* if it is not non-degenerate. Now suppose that $\dim V = n$, prove that if $\omega \in \Lambda^2(V^*)$ is degenerate then the n-fold wedge power of ω vanishes: $\omega^n = 0$.

5. (a) Define immersion and submersion.

   (b) Define embedding.

   (c) Prove that if the domain is a compact manifold then a one-to-one immersion is an embedding.

6. (a) Let $X = y\frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$. So $X, Y \in \mathcal{X}(\mathbb{R}^2)$ are vector fields. Compute the Lie derivative $\mathcal{L}_X Y$ of Y with respect to X.

   (b) Let $X = y\frac{\partial}{\partial x} \in \mathcal{X}(\mathbb{R}^2)$ and $\omega = xdy + ydx \in \Omega^2(\mathbb{R}^2)$. Compute the Lie derivative $\mathcal{L}_X\omega$ of ω with respect to X.