Ph.D. Qualifying Exam in Topology

Ben Cooper, Keiko Kawamuro

January 13, 2023

Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A - Algebraic Topology

1. Let $D = \{z \in \mathbb{C} \mid ||z|| \leq 1, z \neq 0\}$ be the unit disk punctured at the origin.
   (a) Find a path-connected $3:1$ covering of D, and find it's deck transformation group.
   (b) Find a universal covering of X.
2. Let S^3 be the unit sphere in $\mathbb{C}^2$. Let p, q be coprime integers with $0 < q < p$. Let
$$\phi : \mathbb{Z}/p\mathbb{Z} \to \text{Homeo}(S^3)$$
   be a group action defined by:
$$(\phi(1))(z, w) = (e^{2\pi i/p}z, e^{2\pi iq/p}w)$$
   where $(z, w) \in S^3 \subset \mathbb{C}^2$. Compute the fundamental group of the quotient space of the action, $S^3/\sim$.
3. Let D be the unit disk in $\mathbb{R}^2$. Show that every continuous map
$$h : D \to D$$
   has a fixed point.
4. Using van Kampen's Theorem compute the fundamental group of the following spaces.
   (a) An orientable genus 2 surface.
   (b) A Klein bottle.
5. Let $X = S^1 \vee S^1$. Let $x_0 \in X$ be the point where the two circles meet.
   (a) Find a 3:1 normal path-connected covering $p_1 : \tilde{X} \to X$ of X, and compute the normal subgroup $p_{1*}(\pi_1(\tilde{X}, \tilde{x_0}))$ of $\pi_1(X, x_0)$.
   (b) Find a 3:1 non-normal path-connected covering of X.
6. Compute the fundamental group of the complement of a trefoil knot in $\mathbb{R}^3$.

Part B

1. (a) Define homotopy equivalence $X \simeq Y$ of topological spaces.
   (b) Define a vector bundle $\pi : E \to M$
   (c) Assume M is compact. Prove that if $\pi : E \to M$ is a vector bundle then $E \simeq M$.

2. Prove that if $p : N \to M$ is a surjective submersion, then the pull-back map $p^* : \Omega^*(M) \to \Omega^*(N)$ is an injective algebra homomorphism.

3. Prove that if M is a smooth manifold and $\pi_1(M)$ has no non-trivial index 2 subgroup then M is orientable.

4. (a) State Stokes' Theorem on Differential forms
   (b) let $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Calculate $\int_{S^2} \omega$ using Stokes' Theorem, where

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

   (c) Is $\omega|_{S^2} = d\eta$ for some 1-form $\eta \in \Omega^1(S^2)$? Justify your answer.

5. (a) Prove or disprove: every C^∞ function is analytic.
   (b) Prove or disprove: every analytic function is C^∞.

6. (a) Introduce a volume form ω on $\mathbb{R}^n$. Why is your answer correct?
   (b) Calculate $i_X(\omega)$, $di_X(\omega)$ and $\mathcal{L}_X(\omega)$ where

$$X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$