

Ph.D. Qualifying Exam in Topology

Ben Cooper, Keiko Kawamuro

January 13, 2024

Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A - Algebraic Topology

1. Using van Kampen's Theorem, compute the fundamental groups of the following spaces.
 - (a) An orientable genus 2 surface.
 - (b) An orientable genus 3 surface.
2. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a torus, obtained from the square $[0, 1] \times [0, 1]$ gluing the parallel sides. Remove four points $(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$ from T . Denote the resulting 4-punctured torus by Y . Let

$$\varphi : \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Homeo}(Y)$$

be an action of the group $\mathbb{Z}/2\mathbb{Z}$ on Y defined by $\varphi(1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The action induces a covering map $p : Y \rightarrow Y/(\mathbb{Z}/2\mathbb{Z})$.

- (a) Describe the topological type of the quotient space $Y/(\mathbb{Z}/2\mathbb{Z})$.
 - (b) Compute the deck transformation group of the covering $p : Y \rightarrow Y/(\mathbb{Z}/2\mathbb{Z})$.
3. Let Σ be an orientable genus 2 surface. Let α_0 and α_1 be simple closed curves in Σ . Suppose that α_0 and α_1 are not null-homotopic, and there is no continuous family $\{\alpha_t \mid t \in [0, 1]\}$ of simple closed curves in Σ connecting α_0 and α_1 . Let $\pi : \tilde{\Sigma} \rightarrow \Sigma$ be a universal covering. Let $\tilde{\alpha}_0 \subset \pi^{-1}(\alpha_0)$ and $\tilde{\alpha}_1 \subset \pi^{-1}(\alpha_1) \subset \tilde{\Sigma}$ be connected components of the preimages of α_0 and α_1 . Suppose that α_0 and α_1 do not bound any bigons in Σ . Show that $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ do not bound any bigons in $\tilde{\Sigma}$.
4. Let $X = S^1 \vee S^1$. Let $p_0 \in X$ be the base point where the two circles are glued. Let x, y be generators of the free group $\mathbb{Z} * \mathbb{Z} = \pi_1(X, p_0)$ corresponding to the two circles.

- (a) Give a two-sheeted covering $p : (\tilde{X}, \tilde{p}_0) \rightarrow (X, p_0)$, then compute the image $p_*(\pi_1(\tilde{X}, \tilde{p}_0)) < \mathbb{Z} * \mathbb{Z}$.
 - (b) Find a covering $\rho : (Z, z_0) \rightarrow (X, p_0)$ such that

$$\rho_*(\pi_1(Z, z_0)) = \langle x \rangle < \mathbb{Z} * \mathbb{Z}.$$

5. Compute the fundamental group of the complement of a positive trefoil knot in \mathbb{R}^3 (a knot with 3 positive crossings). Do the same for a negative trefoil knot (a knot with 3 negative crossings).
6. Find a 2-dimensional cell complex X whose fundamental group is isomorphic to $\langle a, b \mid a^2 = b \rangle$.

Part B

1. (a) Let $p : \tilde{M} \rightarrow M$ be a covering space of a smooth manifold M . Prove that \tilde{M} is a smooth manifold and p is a smooth map between smooth manifolds.
- (b) A property P is *subgroup-closed* if for all subgroups $H \subseteq G$ of groups G , P is true for G implies that P is true for H . For example, the property: G is abelian, is subgroup-closed.
Fix $n \in \mathbb{Z}$. We say that a group G is n -manifoldy when there exists a smooth path-connected n -dimensional manifold M such that

$$\pi_1(M) \cong G$$

Prove that n -manifoldyness is subgroup-closed.

- (c) Which groups are 1-manifoldy? Which groups are 2-manifoldy?
2. (a) Define orientable smooth manifold.
- (b) Prove or disprove. Let $p : \tilde{M} \rightarrow M$ be a covering space of a smooth manifold M . If M is orientable then \tilde{M} is orientable.
- (c) Prove or disprove. Let $p : \tilde{M} \rightarrow M$ be a covering space of a smooth manifold M . If \tilde{M} is orientable then M is orientable.
3. Prove that an embedding $i : M^m \rightarrow N^n$ of smooth manifolds determines an $(n - m)$ -dimensional vector bundle $p : E \rightarrow M$ on M consisting of the directions normal to M at each point:

$$\forall q \in M, v \in p^{-1}(q) \Rightarrow v \in T_p N \text{ and } v \notin \text{im}(di_p)$$

4. (a) Define the terms closed form and exact form.
- (b) Prove that every closed form $\theta \in \Omega^1(\mathbb{R})$ is exact

- (c) Consider the group action of \mathbb{Z} on \mathbb{R} by translation: if $n \in \mathbb{Z}$ then

$$\phi_n : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is} \quad \phi_n(x) := x + n$$

Let $\Omega^*(\mathbb{R})^{\mathbb{Z}} := \{\theta \in \Omega^*(\mathbb{R}) : \phi_n^*(\theta) = \theta, \text{ for } n \geq 0\}$ be \mathbb{Z} -equivariant forms on \mathbb{R} .

- i. Prove that $\Omega^*(\mathbb{R})^{\mathbb{Z}} \subset \Omega^*(\mathbb{R})$ is a subcomplex.
 - ii. Construct a closed form $\theta \in \Omega^1(\mathbb{R})^{\mathbb{Z}}$ which is not exact. Prove that your answer is correct.
5. (a) Give at least two definitions of the Lie derivative $\mathcal{L}_X Y$ of a vector field Y by a vector field X .
- (b) Construct two vector fields X and Y on \mathbb{R}^2 with $\mathcal{L}_X Y \neq 0$.
- (c) A map $\phi : \Omega^*(M) \rightarrow \Omega^*(M)$ is *nulhomotopic* if

$$\phi = dH + Hd$$

for some $H : \Omega^{*-1}(M) \rightarrow \Omega^*(M)$. Let X be a vectorfield on M . Prove that the Lie derivative

$$\mathcal{L}_X : \Omega^*(M) \rightarrow \Omega^*(M)$$

is nulhomotopic.

6. Let M be a smooth manifold.

- (a)
 - i. Define germs C_p^∞ for $p \in M$
 - ii. The tangent space $T_p M$ in terms of germs C_p^∞ .
 - iii. For a smooth map $F : N \rightarrow M$ define $F_* : T_p N \rightarrow T_{F(p)} M$ in terms of germs.
- (b) If $\gamma : \mathbb{R} \rightarrow M$ is smooth then prove that

$$\gamma_* \left(\frac{d}{dt} \Big|_p \right) = \sum_{i=1}^m \frac{d\gamma^i}{dt}(\gamma(p)) \frac{\partial}{\partial x^i} \Big|_p$$

in a chart (U, x^1, \dots, x^m) about $\gamma(p)$