Simplifying Expressions (Including Exponents and Logarithms)

Math Tutorial Lab Special Topic *

Combining Like Terms

Many times, we'll be working on a problem, and we'll need to simplify an expression by combining like terms. In other words, if multiple terms contain the same variable (raised to the same power), then we want to combine those terms together. Some examples are given below.

Example Simplify $3x + 4y + 6z + 7y + 2x$.

Example Simplify $xy + 8x + 6y + 4xy + 5x$.

Example Simplify $3x + 5x^2 + 2 + 4x^2 + 3$.

Exponential Expressions

An exponential expression has the form a^b, where a is called the base, and b is called the exponent. Remember that $a^b = a \cdot a \cdot a \cdot \ldots \cdot a$, that is, a multiplied by itself b times.

We have several properties of exponential expressions that will be useful. For positive real numbers a, b and rational numbers r, s, we have:

1. $a^0 = 1$
2. $a^{-r} = \frac{1}{a^r}$

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3. $a^{r+s} = a^r a^s$

4. $a^{r-s} = \frac{a^r}{a^s}$

5. $(a^r)^s = a^{rs}$

6. $a^r b^r = (ab)^r$

7. $\frac{a^r}{b^r} = (\frac{a}{b})^r$

Example Simplify $\frac{25(a^3)^3 b^2}{5a^2 b(b^2)}$.

Example Simplify $(xy)^{-2}(\frac{2x^2}{y^2})^4$.

When working with exponential expressions, you will often encounter the number e as a base. Called the natural exponential, $e \approx 2.71828\ldots$. Using e as a base follows all the same rules listed above. Think of e as just another (special) number!

Example Simplify $(e^x)(e^{-x})$.

Example Simplify $e^x + e^{2x} + 3e^x + (e^{-x})(e^{3x})$.

Fractional Exponents and Roots

In the examples above, we worked with whole numbers or variables as our exponents. However, exponents can also be fractions. When we have a fraction as an exponent, we're really taking a root. This can be written as follows. If we have a positive real number a and a rational number $r = \frac{p}{q}$, where $\frac{p}{q}$ is in lowest terms and $q > 0$, then

$$a^r = a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

For example, $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$. Fractional exponents follow the same exponential rules that we have listed above. In addition, from rules 6 and 7 in the list above, we get the following useful rules for dealing with roots. If a, b are a positive real numbers and $q > 0$, then we have:

1. $\sqrt[q]{a}\sqrt[q]{b} = \sqrt[q]{ab}$
2. $\frac{\sqrt[q]{a}}{\sqrt[q]{b}} = \sqrt[q]{\frac{a}{b}}$

Example Simplify $\sqrt{72}$.

Example Simplify $2\sqrt{500x^3}$.

Example Simplify $6^{\frac{1}{2}}(\sqrt[5]{6})^3$.

Example Simplify $(\frac{81}{256})^{-\frac{1}{4}}$.

Example Simplify $(r^{\frac{2}{3}}s^3)^2\sqrt{20r^4s^5}$.

Logarithms

We've spent the last few sections talking about exponents. We'll now shift our focus to logarithms. You can think of a logarithm as the inverse of the exponential. A log is defined as follows: for any positive number $a \neq 1$ and each positive number x, $y = \log_a x$ if and only if $x = a^y$.
Some properties of logarithms are listed below. Assume x, y are positive real numbers, and that r is a real number.

1. $\log_a a^r = r$
2. $a^{\log_a x} = x$
3. $\log_a(xy) = \log_a x + \log_a y$
4. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
5. $\log_a x^r = r \log_a x$

Example Simplify $\log_4 64$.

Example Simplify $\log_8 2$.

Example Simplify $\log_5 5^2$.

Simplify the following logarithms so that the result does not contain logarithms of products, quotients, or powers.

Example $\log_3 \frac{x^4}{x+1}$.

Example $\log_3 \frac{(3x+2)^{\frac{3}{2}}(x-1)^3}{x\sqrt{x+1}}$.

There are two common bases that have special notation. The first, base 10, is typically omitted when writing a logarithm. For the positive number a, this means that $\log_{10} a = \log a$. The other base, base e, corresponds to the natural exponential we saw above. A log with base e is called a **natural log** and is written as follows: $\log_e a = \ln a$. These bases are simply special cases of the logs we've already be studying, so all of the above rules apply.

Example Simplify $\ln e^{\frac{1}{3}}$.

Example Simplify $e^{2\ln \pi}$.

Rewrite the expression as a single logarithm.

Example $\ln(x-1) + \frac{1}{2}\ln x - 2\ln x$.

Example Write $\ln(8) + \ln(\frac{4}{9})$ in terms of $\ln(2)$ and $\ln(3)$.

References

Many ideas and problems inspired by `www.khanacademy.org` and the fifth edition of *PreCalculus* by J. Douglas Faires and James DeFranza.