Ph.D. Qualifying Exam in Topology
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Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A

Let S_g the closed genus g surface with with the standard topology. Let $S^n \subset \mathbb{R}^{n+1}$ be the standard unit sphere.

1. (a) Derive a presentation for $\pi_1(S_2, p)$ where $p \in S_2$ is a point. Be sure to justify the steps of your derivation with clear statements of the theorems you are using, and appropriate drawings.

   (b) Using your presentation from part (a), prove that $\pi_1(S_2, p)$ is not abelian.

2. Let X be a path connected topological space and suppose that $p, q \in X$. Prove that $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$.

3. (a) Suppose that X is path connected, locally path connected, and Hausdorff. Define $p : E \to X$ is a **covering map**. What does it mean for p to be a **regular covering**? What is a **deck transformation**?

   (b) Prove that if $p : (E, e) \to (X, x)$ is a covering map, then the group of deck transformations of p is isomorphic to

   $$N(p_{\#}\pi_1(E, e))/p_{\#}\pi_1(E, e)).$$

   State any theorems you are using in the proof.

4. (a) Give an example of a regular covering of (S_2, p) that is not the universal covering of S_2.

   (b) Give an example of an irregular covering of (S_2, p).

   (c) Describe the universal cover of S_2.

5. Suppose that X is a compact Hausdorff space and that $A, B \subset X$ are closed sets with $A \cap B = \emptyset$. Prove that there exist $U, V \subset X$ open with $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

6. Let $\mathbb{C}P(1)$ be the quotient space obtained from $\mathbb{C}^2 - \{\vec{0}\}$ from the equivalence relation $(z_1, z_2) \sim (w_1, w_2)$ if there exists $\lambda \in \mathbb{C} - \{0\}$ so that $\lambda(z_1, z_2) = (w_1, w_2)$. Denote the equivalence class of (z_1, z_2) by $[z_1, z_2]$.

(a) Prove that $\mathbb{C}P(1)$ is compact, and Hausdorff.

(b) Prove that $\mathbb{C}P(1)$ is homeomorphic to S^2.

Part B

1. Let $x^1, \cdots, x^n$ be the standard coordinate functions on $\mathbb{R}^n$. A function $f \in C^\infty(\mathbb{R}^n)$ is called strictly convex if the Hessian matrix is positive definite everywhere
$$(\frac{\partial^2 f}{\partial x^i \partial x^j}) > 0.$$
Define
$$y^j = \frac{\partial f}{\partial x^j}, \quad j = 1, \cdots, n$$
and
$$g = \sum_{i=1}^{n} x^i y^i - f.$$
   (a) Prove that at each point $\vec{v} \in \mathbb{R}^n$, there is a neighborhood $U$where $(U, \{y^i\})$ is a local coordinate system.

   (b) Prove that as the Hessian of g at any point $\vec{v} \in \mathbb{R}^n$ with respect to local coordinates $y^1, \cdots, y^n$ satisfies the following
$$(\frac{\partial^2 g}{\partial y^i \partial y^j}) = (\frac{\partial^2 f}{\partial x^i \partial x^j})^{-1}.$$

2. Let α be a one-form defined on a smooth three-manifold M such that $\alpha \wedge d\alpha$ is non-zero everywhere.

   Prove that there exists a unique smooth vector field V on M such that
$$\iota_V(d\alpha) = 0, \text{ and } \alpha(V) = 1.$$

3. Let $i : M \to \mathbb{R}^4$ be a smooth embedding, where M is a compact closed smooth orientable manifold of dimension 3. Let $x^0, \cdots, x^3$ be the standard coordinate functions on $\mathbb{R}^4$. Define
$$\omega = i^*(x^0 dx^1 dx^2 dx^3 - x^1 dx^0 dx^2 dx^3 + x^2 dx^0 dx^1 dx^3 - x^3 dx^0 dx^1 dx^2).$$
   Prove that $[\omega] \in H^3(M)$ is non-zero.

4. State without proof the deRham cohomology of S^n. Use it to compute the deRham cohomology of $S^n \times S^m$, where $n \geq m \geq 1$. (Hint: Use induction).

5. Let $H = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = 1\}$.

   (a) Prove that H is a regular submanifold of $\mathbb{R}^3$ and describe the tangent space to H at any point $(a, b, c) \in H$.

   (b) Let $p : H \to \mathbb{R}^2$ be the restriction of $p(x, y, z) = (x, y)$ to H. Where does $p_* : T_{(a,b,c)}H \to T_{(a,b)}\mathbb{R}^2$ have rank 0, 1 and 2. Draw a sketch of the plane where you have indicated where the regular and singular values of $p : H \to \mathbb{R}^2$ lie.

6. Let $SL_2\mathbb{C}$ be the set of 2×2 matrices of determinant 1 lying in $\mathbb{C}^4 = M_2(\mathbb{C})$.

   (a) Prove that $SL_2\mathbb{C}$ is a regular submanifold of $M_2(\mathbb{C})$.

   (b) Describe the tangent space at the identity of $SL_2\mathbb{C}$.

   (c) Prove that $SL_2\mathbb{C}$ is a Lie group.