

Ph.D. Qualifying Exam in Topology Spring 2019

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Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A

- Let X be a Hausdorff space and $S \subset X$ be a compact subspace. Prove that S is closed.
 - Let S be a compact space and S' be a Hausdorff space. Prove that any continuous map $f : S \rightarrow S'$ is a closed map.
- Let S be a connected topological space. Let $f : S \rightarrow \mathbb{R}$ be a continuous map. Suppose that f is locally constant; that is, for any point $a \in S$ there exists a neighborhood U_a of a such that $f(x) = f(a)$ for all $x \in U_a$. Prove that f is a constant function.
- Show that every continuous map $f : D^2 \rightarrow D^2$ has a fixed point.
 - Let $X = S^1 \times D^2$. Does there exist a retraction $r : X \rightarrow \partial X$ to the boundary?
 - Show that \mathbb{R}^n where $n \neq 2$ is not homeomorphic to \mathbb{R}^2 .
- Find the fundamental group of the lens space $L(p, q)$. (The definition of $L(p, q)$ is provided on the bottom of this page.)
 - Find the fundamental group of the Klein bottle K .
 - Find the fundamental group of the n th connected sum of the Klein bottle $K \# \cdots \# K$. Determine whether the fundamental group is abelian or non-abelian.
- Let $X \approx D^2 \setminus \{p, q, r\}$ be a thrice-punctured disk. Find a 2-fold cover \tilde{X} of X , and describe how the covering transformation group acts on \tilde{X} .
- Find a 2-dimensional space whose fundamental group is isomorphic to \mathbb{Z}_n .

Definition

Let $X \approx D^2 \times S^1$ be a solid torus embedded in S^3 . Let $T := \partial X \approx S^1 \times S^1$ and a meridian $\mu := S^1 \times \{\cdot\}$ and a longitude $\lambda := \{\cdot\} \times S^1$. Let $\phi : T \rightarrow T$ be a diffeomorphism whose induced homomorphism $\phi_* : \pi_1(T) \rightarrow \pi_1(T)$ of the fundamental group takes the homotopy class $[\mu]$ to the homotopy class $p[\mu] + q[\lambda]$. The Lens space $L(p, q)$ is a closed manifold obtained from S^3 removing T from S^3 and then gluing back T using the diffeomorphism ϕ .

Part B

- Let M be a smooth manifold.
 - If α and β are two closed forms on M , then prove that $\alpha \wedge \beta$ is closed;
 - If α and β are two exact forms on M , then prove that $\alpha \wedge \beta$ is exact.
 - Find a 3 form α in \mathbb{R}^7 such that $\alpha \wedge d\alpha$ is a volume form of \mathbb{R}^7 .
- Let $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ be a 2-form on \mathbb{R}^{2n} . Let $A \in GL(2n, \mathbb{R})$. Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL(2n, \mathbb{Z})$.
 - Compute the wedge product $\omega^n = \omega \wedge \cdots \wedge \omega$.
 - Show that $A^*\omega = \omega$ if and only if $A^t J A = J$.
 - Show that if $A^*\omega = \omega$ then $\det(A) = 1$.

- Let

$$M = \left\{ \left[\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right] \mid x, y, z \in \mathbb{R} \right\}$$

Notice that M is Lie group under the operations of matrix multiplication and matrix inversion.

- Show that M is not isomorphic to \mathbb{R}^3 as Lie groups.
 - Let $\omega = dx \otimes dx + dy \otimes dy + (dz - xdy) \otimes (dz - xdy)$. Show that ω is invariant under left multiplication.
- Let X be the surface of rotation coming from rotating the vertical line through $(1, 0, 0)$ about the z -axis in \mathbb{R}^3 . Let $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the map $p(x, y, z) = (y, z)$.
 - Prove that X is a regular submanifold of \mathbb{R}^3 .
 - Analyze the rank of the restriction of p to X , that where does $p : X \rightarrow \mathbb{R}^2$ have rank 0, 1 and rank 2?

5. Consider the solid of revolution S obtained by rotating the circle of radius 1 centered at $(0, 2, 0)$ in the (y, z) -plane, about the z -axis.

Giving ∂S the boundary orientation from the orientation on S inherited from the standard orientation of \mathbb{R}^3 evaluate $\int_{\partial S} x dy \wedge dz$ by the easiest method you can think of.

6. Let $n > 0$, and $p \in \mathbb{R}^n$. Recall that

$$T_p \mathbb{R}^n = \{L : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R} \mid L \text{ is } \mathbb{R}\text{-linear, and } L \text{ is a derivation centered at } p\}.$$

From the definition prove that $T_p \mathbb{R}^n$ is an n -dimensional vector space.