Ph.D. Qualifying Exam in Topology
Fall 2017

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Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A

1. Let $\mathbb{R}^3 - \{\vec{0}\}$ be given the subspace topology from $\mathbb{R}^3$ in the standard topology. Let $\mathbb{R}P(2)$ be the quotient space of $\mathbb{R}^3 - \{\vec{0}\}$ under the equivalence relation,
$$(x, y, z) \sim (x', y', z')$$
if there exists $\lambda \in \mathbb{R} - \{0\}$ so that
$$\lambda(x, y, z) = (x', y', z').$$
Denote the equivalence class of (x, y, z) by $[x, y, z]$. Let $F : \mathbb{R}P(2) \to \mathbb{R}^3$ be given by
$$F([x, y, z]) = \frac{(yz, xz, xy)}{x^2 + y^2 + z^2}.$$
   (a) Prove that F is well defined and continuous.
   (b) Is F one-to-one? Justify your answer.
   (c) Let $||(x, y, z)|| = \sqrt{x^2 + y^2 + z^2}$. Does $||F||$ have a maximum and minimum value? Justify your answer.

2. Give an example of a topological space X and a subspace A so that the closure of A is bigger than set of all points that are limits of sequences whose image is contained in A.

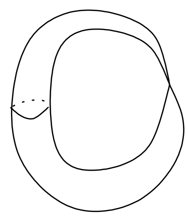
3. Let $\mathbb{R}P(2)$ be the space from the first problem in this set. Compute $\pi_1(\mathbb{R}P(2), x_0)$ for some choice of basepoint. Justify your statements.

4. Classify the following spaces up to homotopy type. That is group them by homotopy type, saying which are homotopy equivalent and which aren't. Justify your answers.

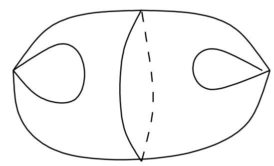
   Recall $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_i x_i^2 = 1\}$ with the subspace topology from the standard topology on $\mathbb{R}^{n+1}$. The **wedge product** of a collection of spaces is the result of choosing a point from each and identifying those points.

   - Let X be the result of identifying $(0, \sqrt{2}, \sqrt{2})$ and $(0, \sqrt{2}, -\sqrt{2})$ in S^2.
   - Let Y be the result of identifying $(0, -\sqrt{2}, \sqrt{2})$ and $(0, -\sqrt{2}, -\sqrt{2})$ in X.

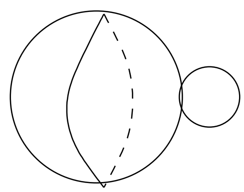
- Let Z be the wedge product of S^2 and S^1.
- Let W be the wedge product of S^2 with two copies of S^1.
- Let the torus T be $S^1 \times S^1$.



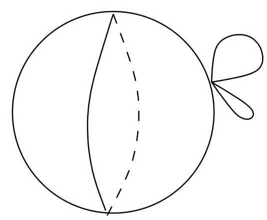
X



Y



Z



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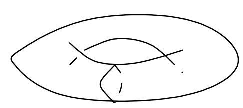

T

5. Let $D^2(r) = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq r^2\}$. Prove that if $f : D^2(1) \to \mathbb{R}$ is continuous then there exists $\overline{f} : D^2(2) \to \mathbb{R}$ continuous so that $\overline{f}|_{D^2(1)} = f$.

6. Prove that a subset of $\mathbb{R}$ in the standard topology is compact if and only if it is closed and bounded.

Part B

1. Let $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ in the standard smooth structure. Here we are viewing $\mathbb{R}^{n^2}$ as $n \times n$-matrices with real coefficients.

   (a) Prove that matrix multiplication,

   $$\mu : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$$

   is a smooth map.

   (b) Prove that the domain of definition of the matrix inverse ι is an open subset of $M_n(\mathbb{R})$, and that on its domain $Gl_n(\mathbb{R})$, the matrix inverse $\iota : Gl_n(\mathbb{R}) \to Gl_n(\mathbb{R})$ is a smooth map.

   (c) Prove that $GL_n(\mathbb{R})$ is a Lie group.

   (d) What is the tangent space at the identity?

   (e) How many connected components does $GL_n(\mathbb{R})$ have?

2. Let $F \subset \mathbb{R}^3$ be a compact, oriented regular submanifold of dimension 2. Suppose that the boundary of F is

   $$\partial F = \{(x, y, z) \in \mathbb{R}^3 | z = 0 \text{ and either } x^2 + y^2 = 1 \text{ or } x^2 + y^2 = 4\}.$$

   Furthermore looking down on ∂F from above the xy-plane the smaller circle is oriented clockwise and the larger circle is oriented counterclockwise. Compute

   $$\int_F dx \wedge dy + dz \wedge dx + dz \wedge dy.$$

3. Going back to the map $F : \mathbb{R}P(2) \to \mathbb{R}^3$ from problem 1 of Part A,

   (a) Display the standard smooth atlas for $\mathbb{R}P(2)$ and check at least one coordinate change is smooth.

   (b) Show that $F : \mathbb{R}P(2) \to \mathbb{R}^3$ is smooth with respect to the standard smooth structures on $\mathbb{R}P(2)$ and $\mathbb{R}^3$.

   (c) Can you find a point $[x, y, z] \in \mathbb{R}P(2)$ where $F_* : T_{[x,y,z]}\mathbb{R}P(2) \to T_{F([x,y,z])}\mathbb{R}^3$ is not injective?

4. Let
$$S^3 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 = 1\}.$$

   (a) Prove that S^3 is a regular submanifold of $\mathbb{R}^4$ in the standard smooth structure on $\mathbb{R}^4$.

   (b) Characterize the tangent space to S^3 at an arbitrary point $(x, y, z, w) \in S^3$.

   (c) Can S^3 be smoothly embedded in $\mathbb{R}^3$ in the standard smooth structure?

5. Let $T^2 = S^1 \times S^1$. Without obsessing to much on smooth structure, compute the De Rham cohomology of T^2. Justify your computation, stating any theorems you are using along the way.

6. Recalling S^3 from Exercise 4.

   (a) Prove that the maps $X(x, y, z, w) = (-y, x, -w, z) \in \mathbb{R}^4$ and $Y(x, y, z, w) = (-z, -w, x, y)$ define smooth vector fields on S^3.

   (b) Compute their Lie bracket. Justify your computation.