Ph.D. Qualifying Exam in Topology

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Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A

Assume $\mathbb{R}^n$ has the usual topology unless otherwise stated.

1. Prove that any open cover of a compact metric space has a Lebesgue number. That is if (X, d) is a compact metric space and $\{U_\alpha\}_{\alpha \in A}$ is a covering of X by open sets, there exists $\epsilon > 0$ so that if $S \subset X$ has $diam(S) < \epsilon$ then there exists α so that $S \subset U_\alpha$.

2. Suppose that X is a compact topological space and Y is a topological space. Give $X \times Y$ the product topology. Suppose that $y \in Y$. Prove that if $X \times \{y\} \subset W \subset X \times Y$ and W is open in $X \times Y$ that there exists $U \subset Y$ open so that

$$X \times \{y\} \subset X \times U \subset W.$$

3. Prove that if $A \subset B \subset \overline{A} \subset X$ where X is a topological space, $\overline{A}$ is the closure of A and A and B have the subspace topology from X. Prove that if A is connected then so is B.

4. Let $X \subset \mathbb{R}^3$ be the union of the unit circle in the xy-plane with the z-axis. Compute the fundamental group of

$$\mathbb{R}^3 - X.$$

   Explain any homotopies you are using with annotated pictures.

5. Let $p : E \to B$ be a covering projection where E and B are path connected, locally path connected and Hausdorff. If $b_0, b_1 \in B$ prove that there is a one-to-one correspondence between $p^{-1}(b_0)$ and $p^{-1}(b_1)$.

6. Let K be a simplicial complex that is a triangulation of $\mathbb{R}P(2)$ the real projective plane.
   (a) Draw such a simplicial complex. I recommend that it be as simple as possible.
   (b) Set up the simplicial chain complex of K, describing bases for the chain groups and boundary maps.
   (c) Compute the homology of the real projective plane.

Part B

1. (a) Define the notion of a vectorfield on a smooth manifold M.

   (b) Define the flow of a vectorfield on a smooth manifold M

   (c) Let X be a vectorfield on M. A *nice periodic orbit* of X is a smooth embedding $\gamma : S^1 \to M$ of an integral curve of X. Construct a vectorfield on the torus $M = S^1 \times S^1$ which has no periodic orbits.

2. Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a smooth embedding. Compute the integral $\int_\gamma \theta$ when

$$\theta = xy^2dx + x^2ydy$$

3. Prove that a covering space of a smooth manifold M is a smooth map between smooth manifolds

4. Suppose that M is a connected smooth manifold and let $p \in M$ be a point. Prove that if $\pi_1(M, p)$ has no non-trivial subgroups of index 2 then M is orientable.

5. A *nulhomotopy* of a map $f : \Omega^*(M) \to \Omega^*(M)$ is a collection of maps

$$H : \Omega^k(M) \to \Omega^{k-1}(M) \qquad \text{where } k \geq 0$$

   which satisfy the equation

$$f = dH + Hd$$

   If X is a smooth vectorfield on M then find a nulhomotopy for the Lie derivative $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ of differential forms with respect to X.

6. Prove that if M is an oriented smooth manifold then the boundary ∂M is also oriented.