Trigonometry Review

Math Tutorial Lab Special Topic*

Right Triangle Trigonometry

We will first define our trig functions using a right triangle. Remember that the angles of a right triangle are no larger than 90°, so we will only consider trig functions restricted to these angles at first. Our trig functions will relate the sides of the right triangle in various ways. With this in mind, we define these functions as follows. For the angle θ in the right triangle shown in Figure 1, we have:

$$\sin\theta = \frac{b}{c},\ \cos\theta = \frac{a}{c},\ \tan\theta = \frac{b}{a},\ \csc\theta = \frac{c}{b},\ \sec\theta = \frac{c}{a},\ \cot\theta = \frac{a}{b}.$$

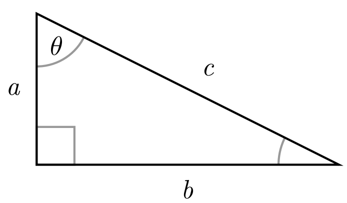


Figure 1

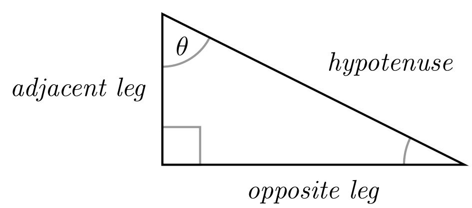


Figure 2

We often use the acronym SOHCAHTOA to remember the formulas for sine, cosine, and tangent. SOHCAHTOA stands for Sine Opposite Hypotenuse, Cosine Adjacent Hypotenuse, Tangent Opposite Adjacent. The terms "opposite," "adjacent," and "hypotenuse" refer to the opposite and adjacent legs and the hypotenuse relative to our angle θ. Remember that the hypotenuse is always the side across from the right angle, and is always the longest side in a right triangle. The acronym SOHCAHTOA tells you the ratio of the sides of the right triangle that give these common trig functions. Figure 2 illustrates this. If we write our trig functions in these terms, we'd have:

$$\sin\theta = \frac{\text{Opposite Leg}}{\text{Hypotenuse}} \qquad \cos\theta = \frac{\text{Adjacent Leg}}{\text{Hypotenuse}} \qquad \tan\theta = \frac{\text{Opposite Leg}}{\text{Adjacent Leg}}$$

$$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite Leg}} \qquad \sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent Leg}} \qquad \cot\theta = \frac{\text{Adjacent Leg}}{\text{Opposite Leg}}$$

We'll now use these formulas to compute the trigonometric functions for some angles in a right triangle. In these examples, you may need to find the third side of the triangle if it's not given. The Pythagorean Formula, written $a^2 + b^2 = c^2$, where a, b, c are the sides of the triangle, with c being the hypotenuse and a, b being the legs, will be useful.

*Created by Maria Gommel, July 2014

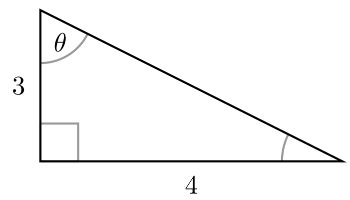


Figure 3

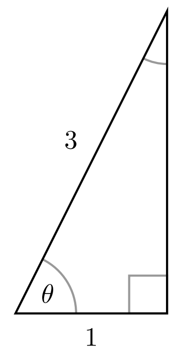


Figure 4

Example Find the value of the six trigonometric functions of the angle θ in Figure 3.

Example Find the value of the six trigonometric functions of the angle θ in Figure 4.

Degrees and Radians

We often learn how to measure angles in degrees. Remember that $360°$ is a full revolution around a circle, $180°$ makes a semicircle, etc. But, in calculus and other advanced math courses, we typically measure angles in **radians**. The **radian measure** of an angle with vertex at the center of a unit circle (a circle with center $(0, 0)$ and radius 1) is the length of the arc made by the angle. We can see this in Figure 5, where the outer part of the circle marked by θ is the radian measure of the angle.

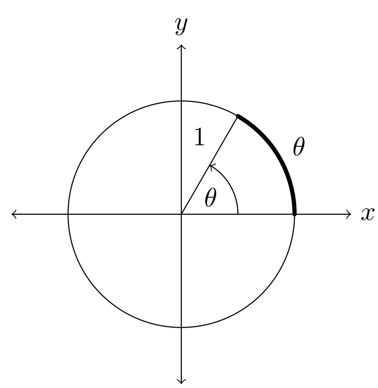


Figure 5

Let's discuss a few common angles. Remember that the circumference of a circle with radius one is 2π. Therefore, an angle of $360°$, will have a radian measure of 2π, since the arc made by a $360°$ angle is the whole circle. Similarly, a $180°$ angle makes a semicircle, so the radian measure is $\frac{2\pi}{2} = \pi$. Below, we list the degree and radian measure of some of the most commonly used angles.

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Based on this reasoning, we also have some general formulas for converting between degree and radian measure. These formulas are as follows:

$$1° = \frac{\pi}{180} \qquad\qquad 1 \text{ radian} = \frac{180°}{\pi}.$$

Example Express $135°$ in radians.

Example Express $\frac{5\pi}{6}$ in degrees.

We introduce one more piece of terminology. The angle θ is **coterminal** with the angle ϕ if θ and ϕ share a terminal side.

Example Find an angle between 0 and 2π that is coterminal to $\frac{23\pi}{6}$.

The Unit Circle and Reference Angles

When we worked with trig functions in the previous sections, we only considered angles that were less than $90°$. But, we'd really like to define our trigonometric functions in a way that allows us to evaluate the functions at any angle. To do this, we make use of the unit circle.

We start by aligning the initial side of our angle with the x axis. Then, we see where the terminal side of our angle crosses the unit circle – call that point (a, b) (see Figure 6). With the terminal side of the angle as our hypotenuse, and the x axis as one leg, we then make a right triangle. If we use this triangle and our definitions of our trig functions on a right triangle, we can see that $a = \cos\theta$ and $b = \sin\theta$. In this way, we define sine and cosine for any angle. Put another way, $\cos\theta$ is the x coordinate where the terminal side of θ intersects the unit circle, and $\sin\theta$ is the y coordinate where the terminal side of θ intersects the unit circle. We then define the other trig functions based on sine and cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \csc\theta = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

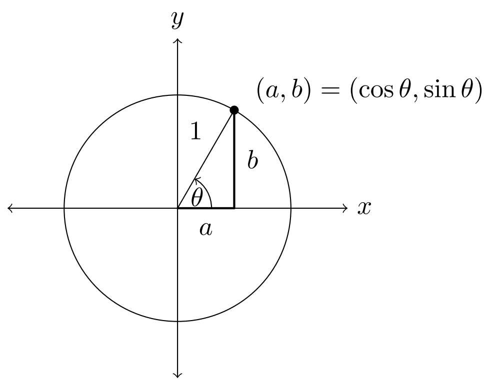


Figure 6

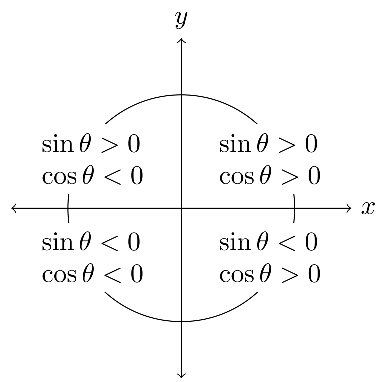


Figure 7

You should notice that these new definitions of the trigonometric functions actually agree with the definitions we had previously for right triangles. Now, however, we can determine the values of our trig functions for any angle.

We have included a unit circle with many common trig values for your reference (see Figure 11). But, instead of memorizing the whole circle, it is often useful to just remember the trig values for angles in the first quadrant. Then, you can make use of reference angles to figure out the value of a trig function for an angle in any other quadrant.

Remember that the **reference angle** for an angle θ is the angle θ makes with the x axis. The reference angle will make it easier to determine trig values for angles that are outside the first quadrant. The reference angle will always be positive.

When graphing an angle in the unit circle, remember that we graph positive angles in the counter-clockwise direction, while negative angles are graphed in the clockwise direction.

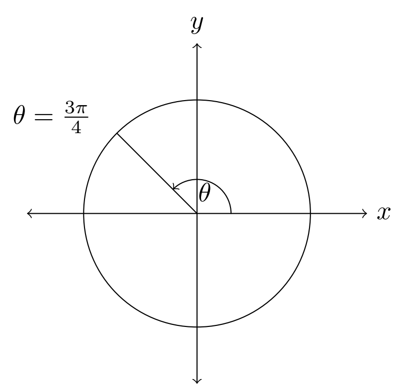
Example Graph $\theta = \frac{5\pi}{4}$ in the unit circle and determine the reference angle.

Example Graph $\theta = -\frac{7\pi}{3}$ in the unit circle and determine the reference angle.

Now, we can use the reference angle and the quadrant that the angle is in to determine the values of sine and cosine (and from these, we can find the values of the other trig functions). In order to determine sine and cosine of θ, we first determine the quadrant that θ is in. This will tell us if sine and cosine of θ are positive or negative. In Figure 7, we show where sine and cosine are positive or negative based on the quadrant. Then, we find the reference angle for θ. The values of $\sin\theta$ and $\cos\theta$ are the same as the values of sine and cosine of the reference angle, but may have a different sign depending on the quadrant θ is in.

Example Determine $\sin\theta$ and $\cos\theta$ for $\theta = \frac{3\pi}{4}$.

- Step 1: We first figure out what quadrant $\theta = \frac{3\pi}{4}$ is in. If we graph this angle, we see that it is in the second quadrant:



- Step 2: If we look at Figure 7, we see that in the second quadrant, $\sin\theta > 0$ and $\cos\theta < 0$. This should make sense intuitively, since for any point (x, y) in the second quadrant, $x < 0$ and $y > 0$. Remember that sine corresponds to the y-coordinate, and cosine corresponds to the x-coordinate.

- Step 3: Now, we determine the reference angle. As we can see from our picture in Step 1, we have a reference angle of $\frac{\pi}{4}$. Remember that $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

- Step 4: We combine our findings from the previous steps. We know that $\sin\theta$ and $\cos\theta$ have the same value as $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$, but the sign is determined by the quadrant that θ is in. So, we have that $\sin\theta = \frac{\sqrt{2}}{2}$ and $\cos\theta = -\frac{\sqrt{2}}{2}$.

Example Determine $\sin\theta$ and $\cos\theta$ for $\theta = \frac{7\pi}{6}$. Use these values to find $\tan\theta, \csc\theta, \sec\theta$, and $\cot\theta$.

Example Determine $\sin\theta$ and $\cos\theta$ for $\theta = \frac{17\pi}{3}$. Use these values to find $\tan\theta, \csc\theta, \sec\theta$, and $\cot\theta$.

Basic Trigonometric Graphs

While graphing trigonometric functions can be quite complicated, we will simply give an introduction to the basic graphs of sine, cosine, and tangent, as well as a few properties of each function.

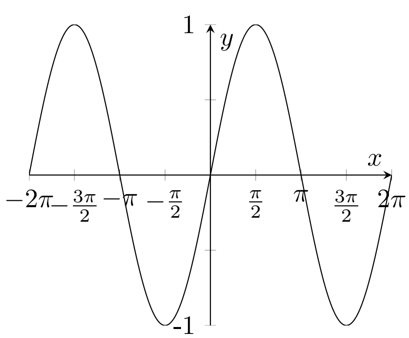


Figure 8: $f(x) = \sin(x)$

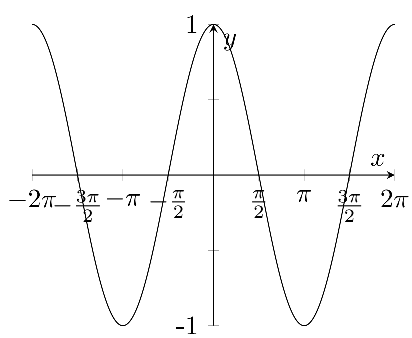


Figure 9: $f(x) = \cos(x)$

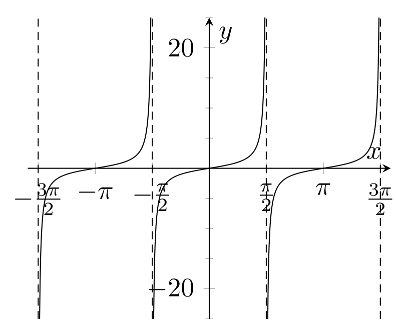


Figure 10: $f(x) = \tan(x)$

The graph of sine is shown in Figure 8. Notice that $\sin(x)$ has a period of 2π, meaning that the graph repeats itself every 2π. Also, observe that the range of $\sin(x)$ is always between -1 and 1.

The graph of cosine is shown in Figure 9. This graph is very similar to that of sine, and has nearly identical properties. The period of $\cos(x)$ is 2π, and the range of $\cos(x)$ is between -1 and 1.

Lastly, the graph of tangent is shown in Figure 10. You'll notice that the graph of tangent is very different than that of sine and cosine. The period of $\tan(x)$ is π, and tangent also has many vertical asymptotes, as indicated by the vertical dotted lines on the graph. As $\tan(x)$ approaches these asymptotes, the graph approaches $\pm\infty$. Thus, the range of $\tan(x)$ is unbounded.

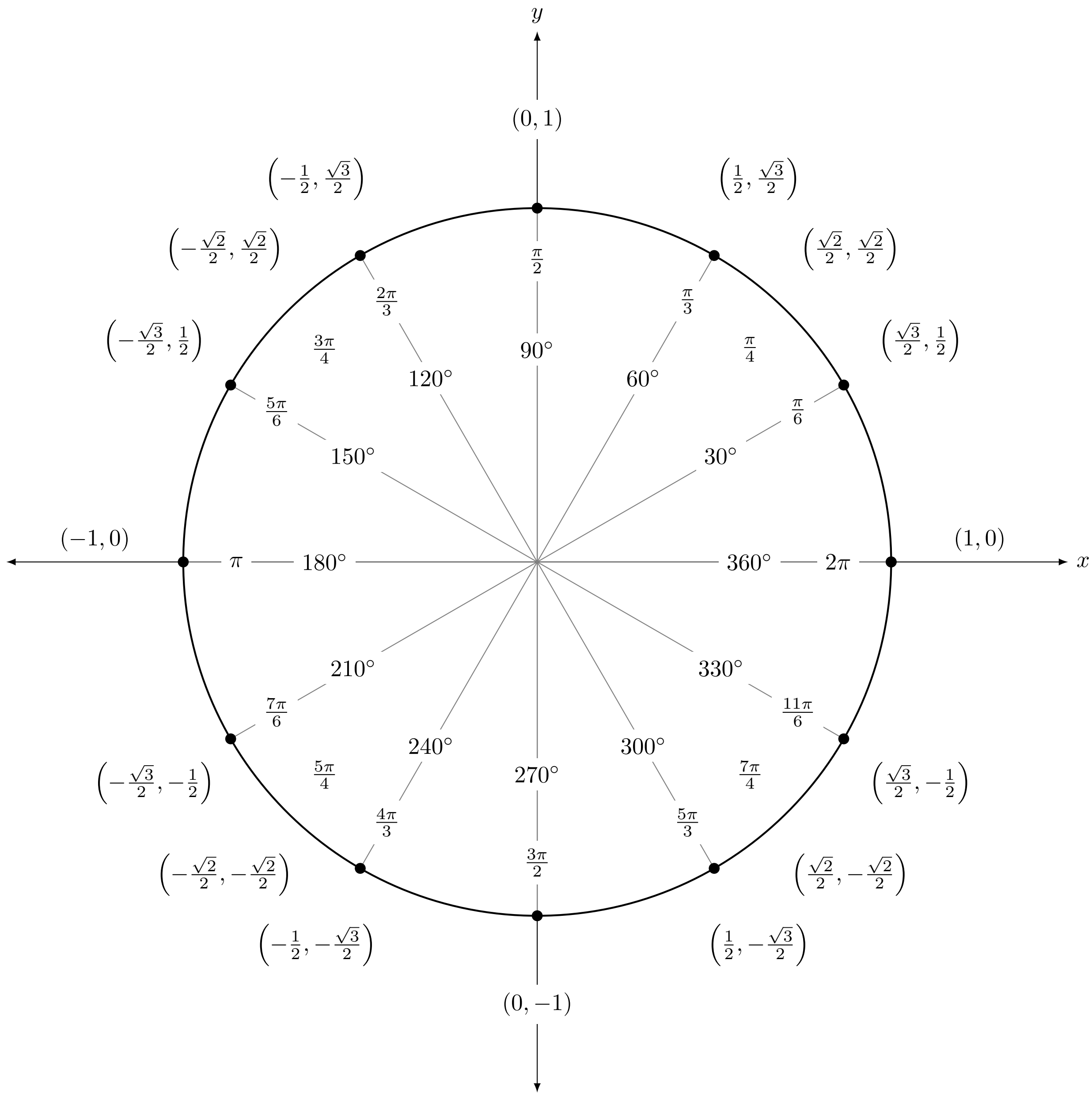


Figure 11

References

Many ideas and problems inspired by `www.khanacademy.org` and the fifth edition of *PreCalculus* by J. Douglas Faires and James DeFranza.

Code for the figures adapted from `http://tex.stackexchange.com/questions/96459/automatically-draw-and-labels-angles-of-a-triangle-in-tikz`, `http://www.texample.net/tikz/examples/tutorial/`, and `http://www.texample.net/tikz/examples/unit-circle/`.