Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

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Instructions: This exam has 4 parts. Do exactly 2 problems from each of the 4 parts. Responses will be judged for correctness, completeness, clarity and orderliness. Justify all statements.

1. Groups:

(1) Let G be a group, and let N be a normal subgroup of G. Suppose that N and G/N are solvable. Prove that G is solvable.

(2) The prime factorization of 2005 is
$$2005 = 5 \cdot 401.$$
Determine all groups of order 2005 up to isomorphism.

(3) Let p be a prime integer.
  (a) Show that any group of order p^n has a non-trivial center.
  (b) Show that any group of order p^2 is abelian.

(4) There are exactly 5 groups of order 27 up to isomorphism, 2 of them non-abelian. More information on the two non-abelian groups is given below. Using this information, classify groups of order $135 = 27 \cdot 5$.

You do not need to use the following information, but we include it for compeleteness: Both $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_9$ admit essentially unique actions of $\mathbb{Z}_3$ by automorphisms, and the two non-abelian groups of order 27 are
$$G_1 = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$$
and
$$G_2 = \mathbb{Z}_9 \rtimes \mathbb{Z}_3.$$

2. Rings:

All rings are assumed to have a multiplicative identity 1.

(1) Let R be a commutative ring and let J_1, J_2 be two ideals of R satisfying $J_1 + J_2 = R$. Given elements $a, b \in R$ prove that there exists $x \in R$ such that
$$x \equiv a \mod J_1 \qquad \text{and} \qquad x \equiv b \mod J_2 .$$

(2) Let R be a commutative ring. Prove that every maximal ideal of R is a prime ideal. What about the converse? Is every prime ideal of R a maximal ideal? (Either prove this or give a counter-example.)

(3) Show that the ring of 3–by–3 matrices over a field is simple.

(4) Let R be any ring and I any ideal. Let n be a natural number. Denote n–by–n matrices over R by $\mathrm{Mat}_n(R)$. Show that $\mathrm{Mat}_n(I)$ is an ideal in $\mathrm{Mat}_n(R)$, and $\mathrm{Mat}_n(R)/\mathrm{Mat}_n(I) \cong \mathrm{Mat}_n(R/I)$.

3. Fields:

Note: A common notation for a field extension $K \supseteq E$ is K/E. This notation is used in exercise (3).

(1) Let K be a field, and let $L \supseteq K$ be the splitting field of a polynomial $f(x) \in K[x]$. Prove that $\mathrm{Aut}_K(L)$ is a finite group.

(2) Let K be a field, and let $L \supseteq K$ be an extension field. Show that $L \supseteq K$ is a finite extension if, and only if, it is finitely generated and algebraic.

(3) Let F be a field of characteristic 0.
  (a) Give an example of extension fields $F \subset E \subset K$ such that E/F is Galois, K/E is Galois, but K/F is not Galois.
  (b) Let K/F be a Galois extension whose Galois group is the symmetric group S_3. Prove that K does not contain a cyclic extension of F of degree 3. How many non-cyclic extensions of degree 3 does K contain? (Recall that a cyclic extension is a Galois extension with cyclic Galois group.)

(4) Define what it means for a real number to be constructible using straightedge and compass. Prove that it is impossible to construct a regular 9-gon with straightedge and compass alone.

4. Linear algebra and modules:

(1) Let D be a division ring such that the center of D contains a field K as a subfield. (Recall that a division ring is a not necessarily commutative ring with multiplicative identity 1 such that every non-zero element is invertible.)
  (a) Show that addition and multiplication in D give D the structure of a vector space over K.
  (b) Assume that D is finite dimensional over K, and let $\alpha \in D$. Prove that there exists a polynomial $f(t) \in K[t]$ of degree ≥ 1 such that $f(\alpha) = 0$.
  (c) Assume that K is algebraically closed, and D is finite dimensional over K. Prove that $D = K$.

(2) Let p be a prime number, and denote the finite field with p elements by $\mathbb{F}_p$, i.e. $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Let S denote the 5–by–5 matrix over $\mathbb{F}_p$ whose entries are equal to 1 except that the entries along the diagonal are all equal to 0,

$$S = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

  (a) Determine the Jordan canonical form of S when $p \neq 5$.
  (b) Determine the Jordan canonical form of S when $p = 5$.

(3) Let K be a field of arbitrary characteristic, and suppose that ζ is a primitive n-th root of unity in K; that is $\zeta^n = 1$, and $\zeta^s \neq 1$ for any $s < n$. (*Note:* You may *not* assume that ζ is a complex n–th root of unity $e^{2k\pi i/n}$.) The goal of this problem is to show that

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0.$$

You are going to use n–by–n matrices over K to show this. Let S denote the n–by–n permutation matrix corresponding to the permutation $(1, 2, 3, \cdots, n)$. For example, for $n = 5$,

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Show that S is similar in $\mathrm{Mat}_n(K)$ to the diagonal matrix D with diagonal entries $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$.

(b) Conclude that S and D have the same trace, and therefore

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0.$$

(4) Let K be a field and let R denote the ring of n–by–n matrices over K. By an R–module, we will mean a unital, left R–module.

(a) Show that every R–module is also a K–vector space.

(b) Show that if two R–modules are isomorphic as R–modules, then they are isomorphic as K–vector spaces.

(c) Show that a finitely generated R–module is free if, and only if, its dimension as a K–vector space is a multiple of n^2.