Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

Fall, 2007

Professors Victor Camillo and Fred Goodman

Instructions: Do exactly two problems from each section for a total of eight problems. Be sure to justify your answers. Good luck.

Notation: For any set A, we denote the cardinality of A by $|A|$. $\mathbb{Z}$ denotes the integers. $\mathbb{Z}_n$ denotes the cyclic group of order n, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers.

1. Groups:

Notation: For any group G, the center of G is denoted by $Z(G)$.

1. Let G be a finite group and H a subgroup. Show directly, without quoting any theorems at all, that the order of H divides the order of G.

2. Let G be a finite group of order p^n where p is a prime. Let H be a normal subgroup of G of size greater than 1. Show that $|H \cap Z(G)| > 1$; that is, H contains a non–identity element of the center of G.

3. Let G be a finite abelian group of order p^n where p is a prime. Show that the following two conditions are equivalent:

(a) G can be generated by two elements, but not by fewer than two elements.

(b) G has a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$, but has no subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$.

4. The prime factorization of 2007 is $2007 = 223 \times 3^2$. Show that any group G of order 2007 has a unique (normal, cyclic) subgroup Q of order 223, and that G is a semi–direct product of Q with a subgroup P of order 9. Show that there exists at least one non–abelian group of order 2007, and classify the abelian groups of order 2007.

2. Rings:

All rings are assumed to have a multiplicative identity 1.

1. Define "Euclidean domain." Show that every ideal in a Euclidean domain is principal.

2. Let R be the ring $\{a + b\sqrt{-5} | a \text{ and} b \text{ are integers}\}$. Show that the ideal in R generated by 3 and $2 + \sqrt{-5}$ is not principal.

3. Define "irreducible" and "prime" elements of a commutative ring multiplicative identity. Show that if R is a principal ideal domain, then every irreducible element is prime.

4. Let R be a commutative ring and let J_1, J_2 be two ideals of R satisfying $J_1 + J_2 = R$. Given elements $a, b \in R$ prove that there exists $x \in R$ such that

$$x \equiv a \mod J_1 \qquad \text{and} \qquad x \equiv b \mod J_2 .$$

3. Fields:

1. Let F be a finite field. Show that F has p^n elements for some prime p. Do this from first principles. Do not assume we know what a prime field is.

2. Let f(x) be a polynomial in Q[x]. Let a be a root of f(x). Show that a is a repeated root of f(x) if and only if a is a root of the derivative of f(x).

3. What is the Galois Group of $x^4 - 1$?

4. Let $f(x)$ be a polynomial over a field K. Show that $f(x)$ has a root in an extension field of K. Do this from first principles; in particular, do not appeal to the existence of splitting fields, as this is the first step in the proof of the existence of splitting fields.

4. Linear algebra and modules:

We will let I denote the identity transformation of a vector space or the identity matrix of any size.

1. Show directly that $\mathbb{Z}_{33} \cong \mathbb{Z}_{11} \oplus \mathbb{Z}_3$ as abelian groups (i.e. $\mathbb{Z}$–modules). Do not appeal to any general theorems; give an explicit isomorphism and prove that it is an isomorphism.

2. Let B denote the matrix

$$B = \begin{pmatrix} -2 & 3 & 0 & 4 \\ -3 & 4 & 0 & 4 \\ -1 & 1 & 5 & 1 \\ -4 & 4 & 0 & 5 \end{pmatrix}.$$

The characteristic polynomial of B is $\chi_B(x) = (x-1)^2(x-5)^2$. Determine the Jordan canonical form of B and find an invertible matrix P such that PBP^{-1} is in Jordan canonical form. *Hint:* One way to do this is first to look for eigenvectors for the two eigenvalues.

3. Suppose a 4–by–4 complex valued matrix A has exactly one eigenvalue λ; that is, the characteristic polynomial of A is $(x-\lambda)^4$. Find the possible Jordan forms for A. Show that $A - \lambda I$ is nilpotent.

4. Let T be a linear transformation on a complex vector space V, not necessarily finite dimensional. Let $\lambda_1, \ldots, \lambda_s$ be distinct eigenvalues of T.

(a) Suppose that for each j $(1 \le j \le s)$ v_j is an eigenvector of T with eigenvalue λ_j. Prove that $\{v_1, \ldots, v_s\}$ is linearly independent.

(b) Now suppose that for each j, v_j is a *generalized eigenvector* of T with eigenvalue λ_j; that is, there is some integer $m_j \ge 1$ such that

$$(T - \lambda_j)^{m_j} v_j = 0.$$

Again conclude that $\{v_1, \ldots, v_s\}$ is linearly independent. (As a matter of notational convenience, assume each m_j is chosen to be minimal; i.e., $(T - \lambda_j)^{m_j - 1} v_j \neq 0$.)