PH.D. QUALIFYING EXAM IN ALGEBRA

Friday, January 13, 2006
Professors Frauke Bleher and Fred Goodman

Instructions: This exam has 4 parts. Do exactly 2 problems from each of the 4 parts. Responses will be judged for correctness, completeness, clarity and orderliness. Justify all statements.

1. GROUPS:

(1) Let Z be the center of a group G, and suppose that G/Z is cyclic. Prove that G is abelian.

(2) Let A be a finite abelian group, written additively, with $|A| = n \geq 2$. For $m \in \mathbb{Z}^+$, define $A_m = \{x \in A \mid mx = 0\}$. If $n = m \cdot m'$, where $m, m' \in \mathbb{Z}^+$ and $\gcd(m, m') = 1$, show that $A = A_m \oplus A_{m'}$. Please do not cite a theorem, but prove this from scratch.

(3) In this exercise you may wish/need to use that the automorphism group of $\mathbb{Z}_7$ is isomorphic to $\mathbb{Z}_6$.
  (a) Show that any group G of order 28 has a normal subgroup N of order 7. Let A denote a 2-Sylow subgroup, of order 4. Show that G is the semi-direct product of N and A.
  (b) Show that there exists a non-abelian group of order 28 with 2-Sylow subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
  (c) Show that there exists a non-abelian group of order 28 with 2-Sylow subgroup isomorphic to Z_4.

(4) (a) Show that for any abelian group, $x \mapsto x^{-1}$ is a group automorphism of order 2. In particular, $\alpha : [x] \mapsto [-x]$ is an automorphism of $\mathbb{Z}_n$ of order 2.
  (b) Show that $\mathbb{Z}_n \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to the dihedral group D_n of order $2n$ (defined as the group of symmetries of the regular n-gon.)
  (c) Determine the center of D_n. Note that the answer is different for n even and odd.

2. RINGS:

All rings are assumed to have a multiplicative identity 1.

(1) Let K be a field.
  (a) Prove that $K[t]$ is a Euclidean domain.
  (b) Prove that every Euclidean domain is a principal ideal domain.

(2) Prove that a commutative ring with identity is a field if, and only if, it is simple.

(3) Let R be a unique factorization domain, and let p be a prime element in R. Let

$$R_{(p)} = \{a/b \mid a, b \in R, p \nmid b\}.$$

Prove that $R_{(p)}$ is a principal ideal domain. Describe all ideals of $R_{(p)}$ and state which ones are maximal.

(4) Let R be an integral domain, let J be an ideal in R, and let p be a nonzero, nonunit element of R.

(a) Show that J is maximal ideal $\implies$ J is a prime ideal.

(b) Show that p is a prime element $\implies$ p is irreducible.

(c) If R is a principal ideal domain, show that p is irreducible $\implies$ pR is a maximal ideal.

(d) Give an example of an integral domain R and an irreducible element p such that pR is not a maximal ideal.

3. Fields:

(1) Let F be a field and let t be transcendental over F (i.e. not algebraic). Let $x \in F(t)$ and suppose $x \notin F$. Write x as a quotient of relatively prime polynomials, $x = f(t)/g(t)$. Prove that $F(t)$ is algebraic over $F(x)$ and express the degree $[F(t) : F(x)]$ in terms of the degrees of $f(t)$ and $g(t)$.

(2) Let F be a field, and let $f(t) \in F[t]$ be a polynomial of degree ≥ 1.

(a) Give the definition of a splitting field of $f(t)$ over F.

(b) Prove that there exists a splitting field of $f(t)$ over F.

(3) Let $f(x) = x^3 - 2$. Prove that $f(x)$ is irreducible over $\mathbb{Q}$. Determine the splitting field K of $f(x)$ over $\mathbb{Q}$ and the degree $[K : \mathbb{Q}]$. Write down all the permutations on the roots of f that are induced by elements of $Gal(K/\mathbb{Q})$.

Determine the isomorphism type of $Gal(K/\mathbb{Q})$.

(4) Let $f(x)$ be an irreducible polynomial of degree n over a field K of characteristic 0. Define the Galois group of $f(x)$ over K. Show that the Galois group of $f(x)$ acts faithfully and transitively on the roots of $f(x)$ in a splitting field. Do not quote any big theorem, such as the fundamental theorem of Galois theory, but prove this from scratch.

4. Linear algebra and modules:

(1) Let V be a finite dimensional vector space over a field K, $V \neq \{0\}$. Let $R = \mathrm{End}_K(V)$. Define a left R-module structure on V by $f\,v = f(v)$ for all $f \in R$ and all $v \in V$. Prove that this makes V into a left R-module, and prove that V is a simple left R-module, i.e. the only R-submodules of V are $\{0\}$ and V.

(2) Let V be a finite dimensional vector space over a field K, $V \neq \{0\}$, and let $A, B : V \to V$ be K-linear maps.

(a) Show that the eigenvalues of AB are the same as the eigenvalues of BA.

(b) Suppose A is invertible and λ is an eigenvalue of A. Prove that $\lambda \neq 0$ and that λ^{-1} is an eigenvalue of A^{-1}.

(3) (a) Determine the possible Jordan canonical forms for a nilpotent 4–by–4 matrix.

(b) Show that if A is a nilpotent n–by–n matrix, then $E + A$ is invertible, where E is the n–by–n identity matrix.

(4) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 1 & 1 & -2 & 3 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\chi_A(x) = (x-1)^2(x+1)(x-3).$$

Find the Jordan canonical form over $\mathbb{C}$ of the matrix A and find an invertible matrix P such that $P^{-1}AP$ is the Jordan form of A.