Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

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Instructions: Do exactly two problems from each section for a total of eight problems. Be sure to justify your answers. Good luck.

1. Groups:

(1) Let G be a group of order 30. Show that G has a normal subgroup of order 15. You may use the Sylow Theorems.

(2) Let G be a finite group. Show that the order of any subgroup of G divides the order of G. Do not quote any theorems, but prove this from scratch.

(3) Suppose T and H are groups and that $\varphi, \varphi' : T \to \mathrm{Aut}(H)$ are homomorphisms. Suppose there is an isomorphism $\alpha : T \to T$ such that

$$\varphi' \circ \alpha = \varphi.$$

Show that the semi-direct products $H \rtimes_\varphi T$ and $H \rtimes_{\varphi'} T$ are isomorphic. (Here H is the normal subgroup in these semi-direct products.)

Hint: Use α to define a set map from the product set $H \times T$ to itself.

(4) Let G be a group (finite or infinite). Prove that if H is a subgroup of finite index n, then G contains a normal subgroup K with $K \leq H$ and $|G : K| \leq n!$.

Hint: Use the natural action of G on the left cosets of H in G.

2. Rings:

All rings are assumed to have a multiplicative identity 1.

(1) An element in a ring R is called irreducible, if whenever $p = xy$, where x and y are elements of R, then one of x or y is a unit. Let R be principal ideal domain and p an irreducible element in R. Show directly that if p divides ab, where a and b are elements of R, then p divides a or p divides b.

(2) Let R be a commutative ring. Recall that an ideal M in R is called maximal if M is not equal to R and if I is an ideal in R with $M \subseteq I \subseteq R$ then $M = R$ or $M = I$. Show that M is a maximal ideal of R if and only if R/M is a field.

(3) Let R be a ring. Recall that an element $x \in R$ is called nilpotent if $x^n = 0$ for some $n \in \mathbb{Z}^+$.

(a) Suppose R is commutative. Prove that the set

$$N(R) = \{x \in R \mid x \text{ nilpotent } \}$$

is an ideal in R. This is called the nilradical of R.

(b) Prove that the elements $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are nilpotent elements in the matrix ring $\mathrm{Mat}_2(\mathbb{Z})$. Prove that $x + y$ is not nilpotent, and deduce that the set of nilpotent elements in $\mathrm{Mat}_2(\mathbb{Z})$ is not an ideal.

(4) Show directly the following variant of Eisenstein's criterion: Let P be a prime ideal in the unique factorization domain R and let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial in $R[x]$ where $n \geq 1$. Suppose $a_n \notin P$, $a_{n-1}, \ldots, a_0 \in P$ and $a_0 \notin P^2$. Prove that $f(x)$ is irreducible in $F[x]$, where F is the fraction field of R.

3. Fields:

(1) Let F be a field and let $g(x)$ be an irreducible polynomial over F. Show directly that there is an extension field of F in which $g(x)$ has a root.

(2) Let $f(x) = x^3 + 2x^2 + 2x + 1 \in \mathbb{Q}[x]$. Determine the Galois group of $f(x)$ over $\mathbb{Q}$. Please show all your work.

(3) Let F be a field of characteristic 0, and let E/F be finite field extension of degree n. Let A be an algebraically closed field containing F. Let $\sigma_1, \ldots, \sigma_n$ be all the distinct embeddings of E over F into A (i.e. σ_i extends the identity on F for all $1 \leq i \leq n$). For $\alpha \in E$, define the **trace** and **norm** of α, respectively, from E to F by

$$\begin{aligned} \mathrm{Tr}_F^E(\alpha) &= \sum_{i=1}^{n} \sigma_i \alpha = \sigma_1 \alpha + \cdots + \sigma_n \alpha \\ \mathrm{N}_F^E(\alpha) &= \prod_{i=1}^{n} \sigma_i \alpha = \sigma_1 \alpha \cdots \sigma_n \alpha \end{aligned}$$

(a) If α is algebraic over F, let

$$p(t) = \mathrm{Irr}(\alpha, F, t) = t^n + a_{n-1} t^{n-1} + \cdots + a_0 .$$

Show that $\mathrm{Tr}_F^{F(\alpha)}(\alpha) = -a_{n-1}$ and $\mathrm{N}_F^{F(\alpha)}(\alpha) = (-1)^n a_0$.

(b) Let E/F be a finite extension with $[E : F] = n$, and let $a \in F$. Determine $\mathrm{Tr}_F^E(a)$ and $\mathrm{N}_F^E(a)$.

(4) Let F be a field of characteristic 0 and let $n \in \mathbb{Z}^+$. Let ζ be a primitive n-th root of unity in some extension field of F, and let $K = F(\zeta)$. Prove that K is Galois and abelian over F.

4. Linear algebra and modules:

All rings are assumed to have a multiplicative identity 1. If M is a left R-module, we assume that $1\,m = m$ for all $m \in M$.

(1) Let R be an integral domain and let M be a module over R. Define $\mathrm{Tor}(M) = \{m \in M \mid am = 0 \text{ for some nonzero element } a \in R\}$. Show that $\mathrm{Tor}(M)$ is a submodule of M.

(2) Let R be a ring, let M be a right R-module, and let A be a right ideal in R. Define

$$X = \left\{ \sum m_k\, a_k \mid m_k \in M \text{ and } a_k \in A \right\}.$$

Here, the sums are finite but may have a different number of non zero terms. Show that X is a submodule of M.

(3) Let R be a ring with 1 and let A_1 and A_2 be left R-modules. Suppose $B_1 \subset A_1$ and $B_2 \subset A_2$ are submodules. Prove that

$$(A_1 \oplus A_2)/(B_1 \oplus B_2) \cong (A_1/B_1) \oplus (A_2/B_2)$$

as R-modules.

(4) Determine the Jordan normal form over $\mathbb{C}$ for the matrix

$$\begin{bmatrix} 0 & 0 & -1 & 2 \\ 2 & -2 & -1 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

and determine a matrix P which conjugates this matrix into its Jordan normal form.