Ph.D. Qualifying Examination in Analysis

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Instructions. Be sure to put your name on each booklet you use.

This examination is a true-false test. Each problem contains a statement that is either true or false. If you believe a statement is true, you must indicate so and give a proof. If you think it is false, you must indicate so and present a counter example.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus.

Part I

1. Let f be a bounded measurable function on the interval $[0,1]$ and let $\epsilon > 0$ be given. Then there is a step function σ on $[0,1]$ such that $|f(x) - \sigma(x)| < \epsilon$ for all $x \in [0,1]$.
2. If $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of nonnegative, Lebesgue integrable functions on $\mathbb{R}$ such that $\{f_n\}_{n\in\mathbb{N}}$ converges to zero pointwise on $\mathbb{R}$, then $\int_{\mathbb{R}} f_n(x)\, dx \to 0$.
3. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of Lebesgue measurable functions defined on $[0,1]$ and for each $n \in \mathbb{N}$ and $x \in [0,1]$, let $F_n(x) = \int_0^x f_n(t)\, dt$. Then the sequence $\{F_n\}_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.
4. The composition of two absolutely continuous functions on $\mathbb{R}$ is absolutely continuous; i.e., if f and g are two absolutely continuous functions defined on $\mathbb{R}$, then $f \circ g$ is absolutely continuous.
5. If f is a continuous, non-decreasing function defined on $[0,1]$ and if $E \subseteq [0,1]$ is a set of Lebesgue measure zero, then $f(E)$ is a set of Lebesgue measure zero.

Part II

6. The equation $\sin(z) = 2$ has no solutions in the complex plane.
7. Suppose f is analytic in a region G (in the complex plane) and that for some positive integer n, the n^{th} derivative of f achieves its maximum modulus at a point z_0 in G. Then f is a polynomial of degree at most n.
8. Let G be a region in the complex plane and let z_0 be a point in G. Suppose that f is a function defined and analytic on $G \setminus \{z_0\}$ and that f maps $G \setminus \{z_0\}$ into the upper half-plane. Then z_0 is a removable singularity of f.
9. The function $f(z) = \csc(z)$ has a simple pole at $z = 0$ and its residue there is 1.
10. Let G be the open upper half-plane and for $z \in G$ define

$$\psi_n(z) := \exp\{\frac{i - (z-n)}{i + (z-n)}\},$$

for $z \in G$ and $n \in \mathbb{N}$. Then $\{\psi_n\}_{n\in\mathbb{N}}$ is a normal family in $H(G)$ with no non-constant limit points.