Ph.D. Qualifying Examination in Analysis

Professors Bor-Luh Lin and Paul S. Muhly

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Instructions. Be sure to put your name on each booklet you use.

Much of this examination is "true-false". When a problem begins with "True-false", you are to decide if the operative assertion is true or false. If you decide that it is "true", you are to give a proof, while if you decide that it is "false", you are to present a counter example.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus.

Part I

1. True-false. Lebesgue measure is *continuous* in the sense that the Lebesgue measure of the closure of a set coincides with the Lebesgue measure of the set.

2. True-false. Let I_1 and I_2 be two disjoint open intervals, and for $i = 1, 2$, let A_i be an arbitrary subset of I_i. Then $m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2)$, where m^* denotes Lebesgue outer measure.

3. True-false. Let $\{f_n\}_{n\geq 0}$ be a sequence of non-negative integrable functions defined on $\mathbb{R}$ such that
$$0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \dots$$
and such that the sequence of numbers $\{\int_{\mathbb{R}} f_n(x)\,dx\}$ is bounded. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Then $f(x) < \infty$ for almost all x.

4. True-false. Let σ be the function on $\mathbb{R}$ that is zero to the left of 1, 1/2 for $1 \leq x < 2$, 3/4 for $2 \leq x < 3$, 7/8 for $3 \leq x < 4$, etc. (Thus, σ jumps by 2^{-n} at n for $n = 1, 2, \dots$, is constant between any two consecutive integers and is continuous from the right.) If $f(x) = x$ on $\mathbb{R}$, then f is integrable with respect to the Lebesgue-Stieltjes measure determined by σ.

5. Let $\{f_n\}$ be a uniformly bounded sequence of measurable functions defined on the interval $[0, 1]$ and let
$$F_n(x) = \int_0^x f_n(t)\,dt \qquad 0 \leq x \leq 1.$$
Show that there is a subsequence $\{F_{n_k}\}$ that converges uniformly on $[0, 1]$.

Part II

1. True-false. Suppose f is analytic in the region $0 < |z| < 1$ and suppose that for each r, $0 < r < 1$, the integral $\int_{C_r} f(z)\,dz = 0$, where C_r is the circle $|z| = r$. Then f is analytic on the open unit disc.

2. Suppose f is analytic in the annular region $1 - \epsilon < |z| < 2 + \epsilon$ for some positive ϵ. Suppose also that $|f| \leq 1$ on the circle $|z| = 1$ and that $|f| \leq 4$ on the circle $|z| = 2$. Show that $|f(z)| \leq |z|^2$ for all z, $1 < |z| < 2$

3. True-false. Let $\mathfrak{G}$ be a domain and let z_0 be a point in $\mathfrak{G}$. Suppose f is analytic in $\mathfrak{G}/\{z_0\}$ and that f takes values in the upper half-plane. Then z_0 is a removable singularity of f.

4. Find the Laurent series representation of the function
$$f(z) = \frac{1}{z^2(1-z)}$$
that is valid in the region $1 < |z| < \infty$.

5. Let f be analytic in the open unit disc $\mathbb{D}$ and let the Taylor series expansion for f be

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Suppose

(a) $f(\mathbb{D}) \subseteq \mathbb{D}$

(b) $a_0 = 0$

(c) $|a_1| = 1$

Calculate $\sup\{|a_n| \mid n \geq 2\}$.