Department of Mathematics University of Iowa Iowa City, IA 52242
117 MH (MacBride) 2:30--5:30 pm, Wednesday, April 12, 2006

MS Exam on Numerical Analysis

Directions: Answer all the 6 problems.

1. Give the Newton method for solving the equation

$$e^x + 4\,e^{-x} - 4 = 0,$$

and discuss the convergence order of the method.

2. Let $f(x) = \sin(\pi x)$. Determine a function $p(x)$ such that $p(x)$ is a polynomial on $[0, 0.5]$ and $[0.5, 1]$, and satisfies the conditions

$$p(x) = f(x),\ p'(x) = f'(x), \quad \text{for } x = 0, 0.5, 1.$$

3. (a) Find A_0, A_1 and A_2 such that the integration rule

$$I(f) = \int_{-h}^{h} f(x)\,dx \approx A_0 f\left(-h/2\right) + A_1 f(0) + A_2 f\left(h/2\right)$$

is exact for polynomials of degree ≤ 2.

(b) Show that the rule constructed in (a) is in fact exact for polynomials of degree ≤ 3.

(c) For the constructed rule, it can be proved that

$$I(f) - \left(A_0 f\left(-h/2\right) + A_1 f(0) + A_2 f\left(h/2\right)\right) = c_0 f^{(4)}(\eta)\, h^5, \quad \eta \in [-h, h]$$

where c_0 is a constant independent of f. Find the constant c_0.

4. Suppose that $B \approx A^{-1}$. Starting with an initial guess x_0, consider the following residual correction method:

$$
\begin{aligned}
&\texttt{for } k = 0, 1, 2, \dots \\
&\quad r_k \leftarrow A x_k - b; \\
&\quad x_{k+1} \leftarrow x_k - B r_k.
\end{aligned}
$$

Show that this will converge (using exact arithmetic) so that $x_k \to x$, with x the exact solution, provided $\|I - BA\| < 1$.

5. What is the QR factorization of a matrix? Explain how a QR factorization of a matrix can be computed using any of the following three methods: (i) Gram–Schmidt orthgonalization, (ii) Givens' rotations, or (iii) Householder reflectors.

   An overdetermined linear system $Ax = b$ where A is $m \times n$ with $m > n$ usually cannot be solved for x. Instead we can ask to minimize $\|Ax - b\|_2$ over all x. Show how the solution to this minimization problem can be computed using the QR factorization of A.

6. Consider the following two methods for numerically solving an initial value problem for the ODE $dx/dt = f(t, x)$:

$$\begin{aligned} x_{n+1} &= x_{n-1} + 2\,h\,f(t_n, x_n) \qquad \text{(leap-frog method)} \\ x_{n+1} &= x_n + (h/2)[f(t_n, x_n) + f(t_{n+1}, x_{n+1})] \qquad \text{(implicit mid-point method)} \end{aligned}$$

where h is the step size, and $t_k = t_0 + k\,h$. For the test equation $dx/dt = \lambda x$, show that the leap-frog method is only stable for $\lambda h = 0$, while the implicit mid-point method is stable for all $\lambda h < 0$.