Department of Mathematics University of Iowa Iowa City, IA 52242

110 MLH 9:00--12:00, Wednesday, August 17, 2005

Numerical Analysis Exam

Directions: Answer all the 6 problems.

1. For a linearly convergent iteration $x_{n+1} = g(x_n)$, g being continuously differentiable, we have $x_{20} = 1.3254943$, $x_{21} = 1.3534339$, $x_{22} = 1.3708962$. Show how to estimate (you do not need to compute the numbers)

   (a) the fixed-point α of the function g;

   (b) the rate of linear convergence;

   (c) the error $\alpha - x_{22}$.

   Hint: From the assumption, there is a constant λ such that for n large, $(x_{n+1}-\alpha)/(x_n-\alpha) \approx \lambda$.

2. Let $f \in C([0,1])$ be given and let $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ be a partition of the interval $[0,1]$. Denote by s the piecewise linear interpolant of f corresponding to the partition; i.e., $s(x)$ is a linear function on each subinterval $[x_{n-1}, x_n]$, $n = 1, \ldots, N$, and $s(x_n) = f(x_n)$, $n = 0, 1, \ldots, N$.

   (a) Give a formula for s on each subinterval.

   (b) Assuming $f \in C^2([0,1])$, bound the error $f(x) - s(x)$.

3. (a) Find the constant c that minimizes $\max_{0\le x\le 1} |e^x - c|$.

   (b) Find the constant c that minimizes $\displaystyle\int_0^1 |e^x - c|^2 dx$.

   (c) Find an equation for the constant c that minimizes $\displaystyle\int_0^1 |e^x - c|\, dx$.

4. Consider solving the initial value problem $y' = f(x, y)$ for $0 \le x \le 1$, $y(0) = Y_0$, f being a smooth function. Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a uniform partition of the interval $[0,1]$ and denote h the step size. For a constant parameter $\theta \in [0,1]$, introduce the following generalized mid-point method

$$y_{n+1} = y_n + h\left[(1-\theta)\, f(x_n, y_n) + \theta\, f(x_{n+1}, y_{n+1})\right].$$

   It is known that for h small enough, this relation defines a unique value y_{n+1}.
   (a) Determine the order of the method.
   (b) Show that the method is absolutely stable when $\theta \in [1/2, 1]$.

5. What is the Cholesky factorization? Find the Cholesky factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{pmatrix}.$$

6. In iteratively solving the linear system $A\boldsymbol{x} = \boldsymbol{b}$ ($\det A \neq 0$), we write $A = P - N$ with P nonsingular, and generate a sequence $\{\boldsymbol{x}^{(k)}\}$ by the formula

$$P\,\boldsymbol{x}^{(k+1)} = \boldsymbol{b} + N\,\boldsymbol{x}^{(k)},$$

starting with some initial guess $\boldsymbol{x}^{(0)}$. Denote the residual $\boldsymbol{r}^{(k)} = \boldsymbol{b} - A\,\boldsymbol{x}^{(k)}$.
(a) Show that the iteration formula can be equivalently expressed as

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + P^{-1}\boldsymbol{r}^{(k)}.$$

(b) Let $\alpha > 0$ be a constant. Define the stationary Richardson method by the formula

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha\,P^{-1}\boldsymbol{r}^{(k)}.$$

Show that the method converges if and only if $\alpha\,|\lambda|^2 < 2\,\mathrm{Re}\,\lambda$ for any eigenvalue λ of $P^{-1}A$.