PhD/MS Qualifying Exam — Numerical Analysis

Department of Mathematics — **University of Iowa**

116 MH — **2:30–5:30 pm, January 23, 2008**

1. Here is some MATLAB^{TM} code to evaluate e^x.

```
function y = myexp(x)
% Computes y = e^x using Taylor series
u = 2.2e-16; % unit roundoff
term = 1;  y = 0;  k = 0;
while abs(term) > u
    y = y + term;  term = x*term/(k+1);
    k = k+1;
end
```

   This code gives accurate values for small x. But when $x = -20$ is used as the input, the computed value is $\approx 5.62 \times 10^{-9}$, while `exp(-20)` gives $\approx 2.06 \times 10^{-9}$; there is not even one correct digit. Investigation shows that during the computation of `myexp(-20)`, the value of `term` becomes as large as $\approx 4.3 \times 10^{7}$. Use your knowledge of floating point arithmetic (in double precision) to explain why the error is so large. This appears to be a problem for large negative values of x. How can you modify this code to give accurate values for all x (ignoring the problems of over- and under-flow)?

2. Write out formulas or pseudo-code for the Newton and secant methods for solving $f(x) = 0$. List all conditions needed in order for these methods to converge. Give the expected rates of convergence of these methods under the conditions you have stated. Explain your terms. Perform three steps of Newton's method to find the positive root of $x^3 - 3x - 3 = 0$ using a starting value of $x_0 = 2$.

3. Given a function $f(x)$ on an interval $[a, b]$ and points $a \le x_0 < x_1 < x_2 < \cdots < x_n \le b$, give a method to construct a polynomial $p(x)$ of degree $\le n$ where p interpolates f at $x_0, x_1, \ldots, x_n$. (You do not have to give pseudo-code, but it should be a clear and complete description.) Give a formula for estimating the interpolation error $f(x) - p(x)$.

What is Chebyshev interpolation? From the formula for the interpolation error, explain how Chebyshev interpolation relates to minimax approximation.

4. Consider solving the initial value problem $y' = f(x, y)$ for $0 \le x \le 1$, $y(0) = Y_0$, f being a smooth function. Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a uniform partition of the interval $[0, 1]$ and denote h the step size. Consider a method of the form

$$\begin{aligned} y_{n+1} &= \alpha\, y_n + \beta\, y_{n-1} + h\, \gamma\, f(x_{n-1}, y_{n-1}), \quad n \ge 1; \\ y_0 &= Y_0, \quad y_1 = Y_0 + h\, f(x_0, Y_0). \end{aligned}$$

Choose the constants α, β, and γ so that the order of the method is as high as possible. Determine whether the resulting method is convergent.

5. Define the Gauss-Jacobi method and the Gauss-Seidel method for solving the linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \in \mathbb{R}^N$ is given and

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -1 \\ & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{N\times N}.$$

For each method, determine a number of iterations to reduce the $\|\cdot\|_\infty$ norm of the error by a factor of 0.01.

6. What is the singular value decomposition (SVD) of a general rectangular matrix $A \in \mathbb{C}^{m\times n}$? For a 3×3 matrix B, compute $\|B\|_2$ and $\mathrm{Cond}_2(B) = \|B\|_2 \|B^{-1}\|_2$ by taking advantage of the following SVD of the matrix B (four significant digits after the decimal point are displayed):

$$\begin{pmatrix} -0.3150 & -0.4193 & -0.8515 \\ -0.6449 & -0.5637 & 0.5161 \\ -0.6963 & 0.7117 & -0.0928 \end{pmatrix} \begin{pmatrix} 10.2377 & 0 & 0 \\ 0 & 4.4830 & 0 \\ 0 & 0 & 0.3050 \end{pmatrix}$$
$$\times \begin{pmatrix} -0.5598 & 0.3230 & 0.7630 \\ -0.7216 & 0.2625 & -0.6406 \\ -0.4073 & -0.9092 & 0.0861 \end{pmatrix}.$$