Qualifying Exam: ODE, Fall, 2008

Please choose 4 out of the 6 problems.

1. In each case, find the value of r at which bifurcations occur, classify types of bifurcations and sketch the bifurcation diagram of fixed points x^* vs r.

(a) $\dfrac{dx}{dt} = r + 2x - x^2;$

(b) $\dfrac{dx}{dt} = rx - x^3.$

2. Consider the flow on a circle given by

$$\frac{d\theta}{dt} = 1 + 2r\cos\theta.$$

(a) Draw a phase portrait on the circle for different cases of the control parameter r.

(b) Find all bifurcation values of r and draw a bifurcation diagram on the $r\theta$-plane.

(c) Compute the oscillation period when the system is an oscillator.

3. Consider the nonlinear system

$$\frac{dx}{dt} = r - x^2, \quad \frac{dy}{dt} = x - y.$$

Assume that $r > 0$.

(a) Find all fixed points and the linearized system at each fixed point.

(b) Find eigenvalues and corresponding eigenvectors for each linearized system.

(c) Classify each fixed point for the linearized system and for the given nonlinear system. Determine their stability.

(d) Sketch a phase portrait of the given nonlinear system.

4. Consider the following model of competition between two species, where $x, y \geq 0$. Find the fixed points, investigate their stability, draw the nullclines and sketch phase portraits. Indicate the basins of attraction of any stable fixed points.

$$\frac{dx}{dt} = x(3 - 2x - y)$$
$$\frac{dy}{dt} = y(2 - x - y).$$

5. Consider the system
$$\frac{d^2x}{dt^2} = x - 4x^3.$$

Find all the equilibrium points and classify them. Find a conserved quantity. Sketch the phase portrait.

6. Show that the system
$$\frac{dx}{dt} = x - y - x(x^2 + y^2)$$
$$\frac{dy}{dt} = x + y - y(2x^2 + y^2)$$

has a periodic solution.

Hint: Rewrite the system in polar coordinates and then construct a trapping region.