Qualifying Examination Spring 2006, Section on Partial Differential Equations

April 5, 2006

(Solve three of the problems.)

1. Solve the initial value problem

$$u_t + xu_x = u$$

   with $u(x,0) = x^2$. Describe and draw the characteristics.

2. Using separation of variables, find the eigenfunctions of the Laplace operator with Dirichlet boundary conditions on the rectangle $[0,\pi] \times [0,2\pi]$.

3. Assume u is twice continuously differentiable on $[0,1]^n \subset R^n$, that u is zero on the boundary of that domain and $|\Delta u| \le 1$. Use the maximum principle to give an estimate of the size of the function u.

4. What is the proper weak solution (i.e. the solution fulfilling the Lax entropy condition) of the equation

$$u_t + u^3 \cdot u_x = 0$$

   for the initial values

$$f_1(x) = \begin{cases} -2 \text{ for } x > 0 \\ 0 \text{ for } x \le 0 \end{cases}$$

   and

$$f_2(x) = \begin{cases} 0 \text{ for } x > 0 \\ -2 \text{ for } x \le 0 \end{cases} \quad ?$$

5. Compute the Fourier series

$$\sum_{k=0} a_k \cos(kx)$$

   for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in [0,\pi/2] \\ -1 & \text{for} \quad x \in (\pi/2,\pi] \end{cases}$$

on the interval $[0,\pi]$. Also solve the heat equation $u_t(x,t) = u_{xx}(x,t)$ on the square $[0,\pi] \times [0,\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary condition $u_x(0,t) = u_x(\pi,t) = 0$. What does it converge to as $t \to \infty$?