# Qualifying Examination on Differential Equations, Fall 2005

August 23, 2005

## 1 Section on ODE

(Solve three of the problems)

1. Find and classify all equilibria of the system of equations

$$\begin{aligned} x' &= x + x^2 + y + y^2, \\ y' &= -x - x^2 + y + y^2. \end{aligned}$$

2. Prove that the system of equations

$$\begin{aligned} x' &= y + x - x^3, \\ y' &= -x + y - y^3 \end{aligned}$$

has at least one non-constant periodic solution. You may assume that $(0, 0)$ is the only equilibrium point.

3. Consider the ode $y' = f(x)$ with a function $f : \mathbb{R} \to \mathbb{R}$ which is infinitely differentiable. Estimate the truncation error for one step of the second-order Taylor method for this equation.

4. Let $y_1$ and $y_2$ be two solutions of the equation $x' = -x^2 + t^2$, and let $y_1(0) = 1, y_2(0) = 2$. Prove that we have $0 < y_1(t) < y_2(t) < y_1(t) + 1$ for all $t > 0$.

## 2 Section on PDE

(Solve three of the problems)

1. Compute the Fourier series

$$\sum_{k=1}^{\infty} a_k \sin(kx)$$

for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in [0, \pi/2] \\ 0 & \text{for} \quad x \in (\pi/2, \pi] \end{cases}$$

on the interval $[0, \pi]$. Also solve the heat equation $u_t(x,t) = u_{xx}(x,t)$ on the square $[0, \pi] \times [0, \infty)$ with the initial value $u(x, 0) = f(x)$ and the boundary condition $u(0, t) = u(\pi, t) = 0$.

2. Using separation of variables, find the eigenfunctions of the Laplace operator with Neumann boundary conditions on the rectangle $[0, 1] \times [0, 1]$.

3. Let $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$. Show that if $u \in C^2(B) \cap C^0(\overline{B})$, $u(x) = 0$ for $|x| = 1$ and $|\Delta u| \leq K$, then also

$$-\frac{K}{2n} \leq u \leq \frac{K}{2n}.$$

Hint: Use maximum principle for a function $v = u - w$ where $w(x) = 0$ for $|x| = 1, \Delta w = \pm K$. Note that $w$ is a simple polynomial.

4. What is the proper weak solution of the equation

$$u_t + u \cdot u_x = 0$$

for the initial values

$$f_1(x) = \begin{cases} 2 \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 \text{ for } x > 0 \\ 3 \text{ for } x \leq 0 \end{cases} ?$$

5. Solve the initial value problem

$$u_t + e^t u_x = u$$

with $u(x, 0) = x$. Describe and draw the characteristics .