Qualifying Examination Fall 2006, Section on Partial Differential Equations

August 23, 2006

(Solve three of the problems.)

1. Solve the initial value problem

$$u_t + 3t^2 u_x = u$$

with $u(x,0) = x^2$. Describe and draw the characteristics.

2. Compute the Fourier series

$$\sum_{k=0}^{\infty} a_k \cos(kx)$$

for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in [0, \pi/2] \\ 0 & \text{for} \quad x \in (\pi/2, \pi] \end{cases}$$

on the interval $[0,\pi]$. Also solve the heat equation $u_t(x,t) = u_{xx}(x,t)$ on the square $[0,\pi] \times [0,\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary condition $u_x(0,t) = u_x(\pi,t) = 0$. What does it converge to as $t \to \infty$?

3. What is the proper weak solution (i.e. the solution fulfilling the Lax entropy condition) of the equation

$$u_t + u^9 \cdot u_x = 0$$

for the initial values

$$f_1(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 1 \text{ for } x > 0 \\ -1 \text{ for } x \leq 0 \end{cases} \; ?$$

4. Assume $u \in C^2\left(\left\{x \in R^3 \mid |x| \leq 1\right\}\right)$, $\Delta u \leq 6$ and $u(x) \geq 0$ for $|x| = 1$. How small can $u(0)$ become ?

5. Using separation of variables, find the eigenfunctions of the Laplace operator with Neumann boundary conditions on the rectangle $[0, 3\pi] \times [0, \pi]$.