Qualifying Exam Topology

August 22, 2005

This is a three hour closed book, closed notes exam. There are two parts to the exam. Work 4 problems from part I and four problems on part II. Indicate clearly on the cover page, which solutions you wish to be graded.

1 Part I

General Topology

1. Prove that every compact Hausdorff space is normal. That is prove that if A and B are disjoint closed subsets of the compact Hausdorff space X, then there are open sets U and V so that $U \cap V = \emptyset$, with $A \subset U$, $B \subset V$.

2. Prove the *Lebesgue number lemma.* That is if X is a compact metric space and $\mathcal{U}$ is an open cover of X, there exists $\epsilon > 0$ so that if D is any subset of X having diameter less than or equal to ϵ then there exists $U \in \mathcal{U}$ with $D \subset U$.

3. Prove that if X and Y are connected topological spaces then $X \times Y$ is connected.

4. Give an example of a space that is connected but not path connected. Justify your answer.

5. Let $\mathbb{R}P(n)$ be the quotient space obtained from $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation, that two points are equivalent if they are scalar multiples of one another. Prove that $\mathbb{R}P(n)$ is second countable, Hausdorff and compact.

6. **a.** Suppose that X is compact and Y is Hausdorff. Prove that every one-to-one, onto, continuous map $f : X \to Y$ is a homeomorphism.

   b. Suppose that X is compact and nonempty, and Y is Hausdorff and connected. Prove that any continuous, open map $f : X \to Y$ is onto.

2 Part II

Differential Topology

1. **a.** Define T_pM where M is a smooth manifold and $p \in M$.

   b. If $F : M \to N$ is a smooth map of smooth manifolds define $F_* : T_pM \to T_{F(p)}N$.

   c. Prove the *chain rule*, that is if $F : M \to N$ and $G : N \to P$ are smooth maps of smooth manifolds then $(G \circ F)_* = G_* \circ F_*$.

2. **a.** Prove that the hyperboloid H of points in $\mathbb{R}^3$ that satisfy

   $$x^2 + y^2 - z^2 = 1$$

   is a smooth submanifold of $\mathbb{R}^3$.

   b. Let $p : H \to \mathbb{R}^2$ be the restriction of orthogonal projection to the xz-plane to H. What are the regular values and critical values of p? Justify your answer.

3. **a.** Define *Lie group.*

   b. Let $SL_n\mathbb{R}$ denote $n \times n$-matrices with real entries of determinant 1. Prove that $SL_n\mathbb{R}$ is a Lie group.

   c. What is the tangent space of $SL_n\mathbb{R}$ at the identity?

4. State and prove the *local immersion theorem.*

5. **a.** If $c : I^n \to M$ is a singular cube, define ∂c.

   b. Give a list of properties of the *exterior derivative* that uniquely determines it on any smooth manifold.

   c. State the *fundamental theorem of calculus.*

   d. Check that the fundamental theorem of calculus is true by evaluating both sides of the equation for the following example. Use the singular cube $c : I^2 \to \mathbb{R}^2$ given by

   $$c(s,t) = (s\cos 2\pi t, s\sin 2\pi t)$$

   and the form $\omega = x\mathrm{d}y - y\mathrm{d}x$. Justify the steps of your computation.

6. **a.** Define *smooth manifold.* On your way you should define all the terms you need for the second part of the question.

   b. Prove that each C^∞-compatible atlas on a manifold is contained in a unique differentiable structure.