Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

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Instructions:

- Do EXACTLY TWO problems from EACH of the four sections.
- Please start a new page for every new problem and put your name on each sheet.
- Please ONLY WRITE on ONE SIDE of each sheet.
- Justify your answers and show your work.
- Please write legibly.
- In answering any part of a question, you may assume the results in previous parts of the SAME question, even if you have not solved them.
- Please turn in the exam questions with your solutions.

Notations:

We adopt standard notations. Namely:

- We write $\mathbb{C}, \mathbb{R}$ and $\mathbb{Q}$ to denote the field of complex numbers, real numbers and rational numbers, respectively; we write $\mathbb{Z}$ to denote the ring of rational integers; when p is a prime number, we write $\mathbb{F}_p$ to denote the finite field with p elements.
- Throughout this exam, R denotes a ring with identity 1; R is called an integral domain if $1 \neq 0$ and R is commutative with no zero divisors.
- All R-modules are assumed to be unital left R-modules.

1 Groups

1. (15 points) Let G be a group and H a normal subgroup. Prove that G is solvable if and only if H and G/H are solvable.

2. Let G be a group acting on a set S, and let $x, y \in S$ be elements in the same orbit.

   (a) (5 points) Prove that the stabilizer subgroups of x and y are conjugate in G.

   (b) (5 points) Recall that $GL_2(\mathbb{F}_q)$ is the group of 2×2 invertible matrices over the field $\mathbb{F}_q$ with q elements, and that $GL_2(\mathbb{F}_q)$ acts on $\mathbb{F}_q^2$ (writing elements as 2×1 column vectors) by left multiplication. Determine the order of the stabilizer of an arbitrary nonzero $v \in \mathbb{F}_q^2$.

   (c) (5 points) Let $S \subset \mathbb{R}^2$ be any collection of 10 distinct points in the plane. Prove the only way a group of order 121 can act on this set is trivially.

3. Let G be an arbitrary group of order $99 = 3^2 \cdot 11$.

   (a) (6 points) Prove that G has a normal subgroup of order 9 and a normal subgroup of order 11.

   (b) (9 points) Prove that every group of order 99 is abelian and determine the number of isomorphism classes of groups of order 99.

2 Rings

1. Let R be a commutative ring, and let S be a multiplicative subset of R in the sense that $1 \in S$ and $xy \in S$ if $x, y \in S$. Define a relation for pairs (a, s) with $a \in R$ and $s \in S$ by

   $$(a, s) \sim (a', s') \quad \text{if} \quad s_1(s'a - sa') = 0$$

   for some $s_1 \in S$.

   (a) (5 points) Prove that this relation is an equivalence relation. Denote by $\frac{a}{s}$ the equivalance class containing (a, s). Denote by $S^{-1}R$ the set of all such equivalence classes.

   (b) (4 points) Define addition in $S^{-1}R$ by

   $$\frac{a}{s} + \frac{a'}{s'} = \frac{s'a + sa'}{ss'}.$$

   Prove that this addition is well defined.

   (c) (4 points) Define multiplication in $S^{-1}R$ by

   $$\frac{a}{s}\frac{a'}{s'} = \frac{aa'}{ss'}.$$

   Prove that this multiplication is well defined.

   (d) (2 points) Prove that $S^{-1}R$ is a ring under the operations above.

2. (15 points) Let R be a commutative ring with $1 \neq 0$. A prime ideal $P \subset R$ is a minimal prime ideal if for all prime ideals $Q \subseteq P$, we have $P = Q$. Prove that a minimal prime ideal of R exists.

3. Let R be a commutative ring with 1. Call ideals I, J relatively prime if $I + J = R$. Prove the following two statements independently (i.e., statement (i) is not used to prove statement (ii)).

   (a) (6 points) Prove that if I and J are relatively prime, then so are I^m and J^n for any positive integers m, n.

   (b) (9 points) Assume I, J are relatively prime and $I \cap J = 0$. Prove that $R \simeq R/I \times R/J$.

3 Linear Algebra and Module Theory

1. Let R be a ring with $1 \neq 0$, and let X and Y be two R-modules. Denote by $\mathrm{Hom}_R(X, Y)$ the set of R-homomorphisms of X into Y.

   (a) (5 points) Define addition in $\mathrm{Hom}_R(X, Y)$ as the addition of mappings into an abelian group. That is, for $f, f' \in \mathrm{Hom}_R(X, Y)$, $(f + f')(x) = f(x) + f'(x)$ for $x \in X$. Prove that $\mathrm{Hom}_R(X, Y)$ is then an abelian group.

   For $a \in R$ and $f \in \mathrm{Hom}_R(X, Y)$, define af by $(af)(x) = a\big(f(x)\big)$ for $x \in X$.

   (b) (5 points) Assume that R is a commutative ring. Prove that $\mathrm{Hom}_R(X, Y)$ is an R-module with $f + f'$ and af as above.

   (c) (5 points) Now assume that R is NOT commutative. Determine whether $af \in \mathrm{Hom}_R(X, Y)$ and whether $\mathrm{Hom}_R(X, Y)$ is an R-module. Why or why not?

2. (15 points) Consider a homomorphism of short exact sequences of modules over a ring as below.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

   Prove that if α and γ are both injective, then so is β.

3. Let $A \in M_n(\mathbb{C})$ be a matrix representing a projection operator on $\mathbb{C}^n$, meaning $A^2 = A$, and let $r = \operatorname{rank} A$.

   (a) (5 points) Prove that the only eigenvalues of A are 0 and 1.

   (b) (5 points) Describe the Jordan canonical form of A (with justification).

   (c) (5 points) Let V be the $\mathbb{C}[x]$-module corresponding to the matrix A. Describe the elementary divisor decomposition of V as a $\mathbb{C}[x]$-module.

4 Field Theory and Galois Theory

1. Consider the field $F = \mathbb{Q}(\sqrt[6]{2})$, where $\sqrt[6]{2}$ is the positive 6^{th} root of 2.

   (a) (4 points) Determine the smallest normal extension K of $\mathbb{Q}$ containing F.

   (b) (4 points) Determine the smallest subfield E of F such that F is normal over E.

   (c) (7 points) Determine the Galois groups $\mathrm{Gal}(K/\mathbb{Q})$, $\mathrm{Gal}(K/E)$, $\mathrm{Gal}(K/F)$, and $\mathrm{Gal}(F/E)$.

2. Let F be a field.

   (a) (2 points) Define the characteristic of F.

   (b) (5 points) Prove that the characteristic of F is either 0 or a positive prime integer.

   (c) (8 points) Let F be a field with a finite number of elements. Prove that $|F| = p^n$ for some prime integer p and some positive integer n.

3. (15 points) Let F be a field and $\alpha \in K$, some extension of F, with α algebraic over F. Prove that there exists a unique monic, irreducible polynomial $m(x) \in F[x]$ such that α is a root of $m(x)$.