**Math@Iowa Analysis Qualifying Exam**
**Friday, August 18, 2023**
**9:00 am - 12:00 noon in Room 210 MLH**

Examination Committee: Professors Ionuţ Chifan and Raúl E. Curto

**Your Name**:

**Instructions**

1. This is a closed-book exam; no books or notes are allowed. This exam includes a Prep Sheet (attached), which you may use as a reference.
2. You are not allowed to use any internet resources; use of calculators is not allowed.
3. This Qualifying Exam consists of 10 questions, broken down into two parts of 5 questions each, for a maximum score of 70 points. Please submit a total of seven question, with at least three chosen from each Part. Please summarize your choices in the table below, by placing a ✓ inside the box located next to each choice.
5. Please write your answers within the space provided for each question. If at all possible, please write your answers in ink (blue, black or red). However, if writing math using a pencil is what you always do, by all means use a pencil for this exam too.
6. For each solution you provide, make sure to include all the details, and state correctly all the known results you use. If appropriate, feel free to refer to the attached Prep Sheet.
7. A score in the range **63 − 70** guarantees a **Ph.D. Pass**; a score in the range **56 − 62** guarantees an **M.S. Conditional Pass**.
8. **Before you start, please write your name on the space provided in this page. When you finish, please turn in this exam booklet.**

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|---|---|---|---|---|
| ☐ R-1A | ☐ R-1B | | ☐ C-1 | |
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| ☐ R-2 | | | ☐ C-2A | ☐ C-2B |
| | | | | |
| ☐ R-3A | ☐ R-3B | | ☐ C-3 | |
| | | | | |
| ☐ R-4 | | | ☐ C-4A | ☐ C-4B |
| | | | | |
| ☐ R-5A | ☐ R-5B | | ☐ C-5A | ☐ C-5B |

# PART I: Real Analysis

(10 points) **(R-1)**

Choose either Problem R-1A or Problem R-1B below and solve it.

**Problem R-1A**. Let $K$ be the set obtained by intersecting the Cantor set $C \subseteq [0,1]$ with the closed interval $[\frac{1}{6}, \frac{2}{3}]$. Prove that $K$ and $C$ are equipotent, by finding an explicit formula for a bijective function $f : K \to C$. Your answer must include a detailed proof of the injectivity and surjectivity of $f$.

**Problem R-1B**. Determine if the following statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

(True or False) Let $\mathcal{CS}$ be the collection of all closed subsets of $\mathbb{R}$. Then $\mathcal{CS}$ is equipotent to $\mathbb{R}$; that is, there exists a bijection between $\mathcal{CS}$ and $\mathbb{R}$.

**Problem R-1 cont.**

(10 points) **(R-2)** On $E := (0, +\infty)$, let

$$f(x) := \frac{1}{x^2}$$

and, for all $n \in \mathbb{N}$, let

$$f_n(x) := \min\left(n^2, f(x)\right).$$

Prove that $f_n \to f$ in measure on $E$.

**Problem R-2 cont.**

(10 points) **(R-3)**

Choose either Problem R-3A or Problem R-3B below and solve it.

**Problem R-3A**. Let $E := [0,1]$ and let $f \in L^2(E)$ such that $\int_E f = 0$ and $\int_E f^2 = 1$. Prove that

$$m(\{f > t\}) \leq \frac{1}{1+t^2},$$

for all $0 \leq t \leq 1$.

**Problem R-3B**. Let $a, b \in \mathbb{R}$, and let $f$ be an absolutely continuous function on $[a,b]$. Assume that there exists a constant $M > 0$ such that

$$|f'| \leq M$$

almost everywhere on $[a,b]$. Prove that $f$ is Lipschitz on $[a,b]$.

**Problem R-3 cont.**

(10 points) **(R-4)** Let $f$ be a uniformly continuous, integrable function on $[0, \infty)$.

(a) Prove that

$$\lim_{x \to \infty} f(x) = 0.$$

(b) Is the conclusion in (a) still true if $f$ is merely continuous instead of uniformly continuous?

**Problem R-4 cont.**

(10 points) **(R-5)**

Choose either Problem R-5A or Problem R-5B below and solve it.

**Problem R-5A**. Determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

A left half-open interval in $\mathbb{R}$ is an interval of the form $(a, b]$ or $(a, +\infty)$, where $-\infty \leq a < b$. Let $\mathcal{F}$ be the collection of left half-open intervals in $\mathbb{R}$ and let $\mathcal{A}(\mathcal{F})$ be the algebra of sets generated by $\mathcal{F}$, that is, the collection of **finite** unions of intervals in $\mathcal{F}$.

(True or False) $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{A}(\mathcal{F})$.

**Problem R-5B**. Let $E$ be a measurable subset of $\mathbb{R}$, let $f \in L^1(E)$, and let $\varepsilon > 0$ be given. Prove that there is a measurable set $A \subseteq E$ with $m(A) < \infty$ such that

(a) $\sup_{x \in A} |f(x)| < \infty$

and

(b) $\int_{A^c} |f| \leq \varepsilon$.

**Problem R-5 cont.**

# PART II: Complex Analysis

(10 points) **(C-1)** Use complex analysis methods to show that

$$\int_{-\infty}^{\infty} \frac{1}{\left(x^2+x+1\right)^2}\, dx = \frac{4\pi}{3\sqrt{3}}.$$

**Problem C-1 cont.**

(10 points) **(C-2)**

Choose either Problem C-2A or Problem C-2B below and solve it.

In Problems C-2A and C-2B, determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

**Problem C-2A**. (True or False) The function

$$\frac{z^2(z-1)}{(1-\cos z)\log(1+z)}$$

has a simple pole at $z = 0$.

**Problem C-2B**. (True or False) There exists a polynomial $p$ with $p(0) \neq 0$, such that, when restricted to the open unit disc $\mathbb{D}$, $p$ establishes a bijection between $\mathbb{D}$ and $\mathbb{D}$. In short, $p|_{\mathbb{D}}$ is a conformal map of $\mathbb{D}$ onto itself.

**Problem C-2 cont.**

(10 points) **(C-3)** (A startling application of Runge's Theorem) On $\mathbb{C}$, and for a given positive integer $n$, consider the set

$$K_n := [\bar{D}(0,n) \setminus \{z \in \mathbb{C} : \mathrm{dist}(z,[0,n]) < \frac{1}{n}\}] \cup \{0\} \cup [\frac{1}{n}, n].$$

(a) Prove that for each $n \geq 1$, $K_n$ is compact.

(b) Prove that for each $n \geq 1$, $\mathbb{C} \setminus K_n$ is connected.

(c) For each $n \geq 1$, let $f_n$ denote the continuous function on $K_n$ with $f_n(0) := 1$ and $f_n(z) := 0$ whenever $z \in K_n \setminus \{0\}$. Fix $n \geq 1$.

Use Runge's Theorem to find a polynomial $p_n$ such that

$$\|p_n - f_n\| := \sup_{z \in K_n} |p_n(z) - f_n(z)| \leq \frac{1}{n}.$$

(d) Show that the polynomials $p_n$ are such that

$$\lim_n p_n(0) = 1,$$

and, for any fixed complex number $z \neq 0$,

$$\lim_n p_n(z) = 0.$$

In other words, the sequence of polynomials $p_n$ converges pointwise to the characteristic function of the singleton $\{0\}$.

**Problem C-3 cont.**

(10 points) **(C-4)**

Choose either Problem C-4A or Problem C-4B below and solve it.

**Problem C-4A**. Suppose that $\varphi$ is analytic at $z_0$ with $\varphi'(z_0) \neq 0$. Also assume that $f$ has a simple pole at $\varphi(z_0)$ with $\operatorname{Res}(f; \varphi(z_0)) = A$. Show that $g = f \circ \varphi$ has a simple pole at $z_0$ and evaluate $\operatorname{Res}(g; z_0)$.

**Problem C-4B**. On the space $H(\mathbb{D})$ of analytic functions on the open unit disc, consider the Hilbert and Cèsaro transforms, defined as

$$H(f)(z) := \int_0^1 \frac{f(t)}{1 - tz}\, dt \quad (f \in H(\mathbb{D}), z \in \mathbb{D})$$

and

$$C(f)(z) := \frac{1}{z} \cdot \int_0^z \frac{f(t)}{1 - t}\, dt \quad (f \in H(\mathbb{D}), 0 \neq z \in \mathbb{D}).$$

(a) Given the sequence of functions $e_n(z) := z^n \quad (z \in \mathbb{D})$, where $n$ is a positive integer, prove that

$$C(e_n) = e_n H(e_n) \quad (\text{for all } n \geq 1).$$

(b) Use the above identity to show that for every $n \geq 1$, the function $C(e_n)$ has a removable singularity at $z = 0$.

**Problem C-4 cont.**

(10 points) **(C-5)**

Choose either Problem C-5A or Problem C-5B below and solve it.

**Problem C-5A**. Determine if the following statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

(True or False). Let $\lambda > 1$. The equation $ze^{\lambda - z} = 1$ has exactly one solution $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(Hint. Use Rouché's theorem, with $f(z) = ze^{\lambda - z}$ and $g(z) = 1 - ze^{\lambda - z}$.)

**Problem C-5B**. Let

$$f(z) = \frac{1}{z(z-1)^2} \quad (z \neq 0, z \neq 1).$$

Find the Laurent Expansion of $f$, in powers of $z$, in the annulus $\operatorname{ann}(0; 1, \infty)$.

**Problem C-5 cont.**

**Additional Space**

**End of Exam**