Numerical Analysis PhD/MS Qualifying Exam

University of Iowa

August 17, 2023

You must answer at least five out of the eight problems. Your score will be based on your best five answers. You must show all work for full credit. You need 80% for a PhD pass and 70% for an MS pass.

Problem 1. Assuming Newton's Method converges and that the initial guess is close enough, use a Taylor Polynomial with error term to show quadratic convergence in the case that $f'(x^*) \neq 0$.

Problem 2. Construct the minimax polynomial $p_2(x)$ on the interval $[-1, 2]$ for $f(x) = |x|$.

Problem 3. The three-point Gaussian quadrature formula for $\int_{-1}^{1} f(x)\, dx$ is

$$I_3(x) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

Determine its degree of precision.

Problem 4. Use Gram-Schmidt orthogonalization to construct three orthogonal polynomials $(\varphi_0, \varphi_1, \varphi_2)$ of degrees 0, 1, 2, on the interval $[-1, 1]$.

Problem 5. Derive the stability function/symbol for the Trapezoid Rule Method for first order initial value problems.

Problem 6. Show that if the graph of a symmetric matrix is a tree (a connected undirected graph with no cycles) then the matrix can be reordered so that the Cholesky factorization gives no fill in.

Problem 7. Let $M \in \mathbb{R}^{n \times n}$ be positive definite and let L be its Cholesky factor so that $M = LL^T$. Show that $\|M\|_2 = \|L\|_2^2$ and $\kappa_2(M) = \kappa_2(L)^2$, where κ_2 is the condition number in the 2-norm.

Problem 8. Derive the numerical method obtained when applying Richardson extrapolation to backward Euler for the problem $u'(t) = f(t, u(t))$ with an initial condition. You will need to recall the asymptotic error of backward Euler.