Qualifying Exam for Math 5600

August 16, 2023

Dr. Zahra Aminzare

INSTRUCTION:

- Please write your full **NAME**:

- The questions for this exam (Math 5600) are divided into two parts.

Answer both questions in Part I.

Answer only one question in Part II.

- If you work on more than one question in Part II, please state **clearly** which one should be graded.
- No additional credit will be given for more than one of the questions in Part II.
- If no choice between the questions is indicated, then the first optional question attempted will be the only one graded.
- All the questions have equal points.

Good Luck!

Part I. Please answer BOTH Question 1 and Question 2.

Question 1. Consider the following system

$$\dot{x} = y + \mu x$$
$$\dot{y} = -y + \mu x - x^2$$

a. Compute a family of center manifolds of the origin which depend on μ.

b. Derive the reduced systems on the center manifolds.

c. Describe any bifurcations that occur as μ changes and draw the corresponding bifurcation diagrams.

Question 2. Consider the following system

$$\dot{x} = -2x(x-1)(2x-1),$$
$$\dot{y} = -2y.$$

a. Use an appropriate Lyapunov function to show that the origin is an asymptotically stable fixed point.

b. Find the basin of attraction, i.e., the set of all points (x, y) such that $\phi_t(x, y) \to (0, 0)$ as $t \to \infty$, where $\phi_t(x, y)$ is the solution of the system starting from (x, y).

c. Does the system admit a periodic solution? Prove it.

d. Sketch the phase plane which qualitatively describes the full dynamics of the system.

Part II. Please answer <u>ONLY ONE</u> of the following questions.

Question 3. Consider the system

$$\dot{x} = x(1-r) - y$$
$$\dot{y} = y(1-r) + x,$$

where $r^2 = x^2 + y^2$. This system has a periodic solution $\gamma(t) = (\cos(t), \sin(t))^\top$. Compute the Poincaré map or Floquet multipliers of $\gamma(t)$ to determine its stability.
Hint: Polar coordinate transformation may help.

Question 4. Consider the system

$$\dot{x} = -y + x(r^4 - 3r^2 + 1)$$
$$\dot{y} = x + y(r^4 - 3r^2 + 1),$$

where $r^2 = x^2 + y^2$. Use the Poincaré-Bendixson Theorem to show that there exist a periodic solution inside $r = 1$ and another periodic solution inside the annular region $1 < r < 2$.

Hint: You may use that $(0, 0)$ is a source without proving it. You may use Poincaré-Bendixson Theorem for an $\alpha-$limit set or an $\omega-$limit set.