Ph.D. and M.S. Qualifying Exam in Algebra

August 20, 9:00am-12:00pm, in room 118 MLH

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Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains 4 sections (group theory, ring theory, linear algebra and module theory, and field theory), each consisting of 4 problems. Please submit solutions for exactly 2 problems, at your choice, from each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show all your work; any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- During the exam you are not allowed to use a calculator, cell phone, ipad, laptop, computer or any other electronic or internet browsing device.

Good luck!

NAME (*PRINT*): ______________________________

Mark in the first column below which problems should be graded!

Your Choice Problem	**Points**	**Your Score**
	20	
	20	
	20	
	20	
	20	
	20	
	20	
	20	
Total	160	

Group theory

G - I: Show that there is no simple group with 616 elements. Include all details and state clearly all theorems/results you use in your proof!

G - II: Assume that G is a finite group that acts transitively on a set X with at least two elements. Show that one can find $g \in G$ such that $g \cdot x \neq x$ for all $x \in X$.
Make sure you include all the details and also state clearly all theorems/results you use in the proof.

G - III: Let G be any infinite group which is finitely generated and admits a subgroup of index 2024. Show that there exists a nontrivial, proper subgroup $H < G$ (i.e. $H \neq 1$ and $H \neq G$) that is characteristic.
Make sure you include all the details in your answer!

G - IV: Let G be a finite nilpotent group such that G/G' is cyclic. Then show that G is cyclic. Make sure you include all the details in your answer!

Ring theory

R - I: Let $R = \mathbb{Z}[\sqrt{-5}]$ and consider the ideals $I = (2, 1+\sqrt{-5})$, $J = (3, 2+\sqrt{-5})$ and $K = (3, 2-\sqrt{-5})$.

1. Prove that $2, 3, 1-\sqrt{-5}, 1+\sqrt{-5}$ are irreducibles in R, no two of which are associate in R and that $6 = 2\cdot 3 = (1+\sqrt{-5})\cdot(1-\sqrt{-5})$ are two distinct factorizations of 6 into irreducibles of R.
2. Prove that I, J, K are non-principal, prime ideals of R.
3. Prove that $IJ = (1-\sqrt{-5})$ and $JK = (1+\sqrt{-5})$. (This shows that products of non-principal ideals can be principal ideal.) Also show that we have the factorizations $(6) = (2)(3) = (1+\sqrt{-5})(1-\sqrt{-5}) = IJ^2K$.

Make sure you include all the details and also state clearly all the results you use in the proof.

R - II:

1. State Eisenstein's irreducibility criterion for primitive polynomials over a UFD.
2. Decide whether the following two polynomials are irreducible or not over $\mathbb{Z}[x]$.
   - $P(x) = x^{277} + 277x^{276} + 277x^{275} + 555$
   - $Q(x) = (x^2+1)^n + 7$ for some positive integer n.

R - III: Let R be an integral domain such that all its prime ideals are principal. Then show that R is a PID. Make sure you prove all the details in your answer!

R - IV: A (not-necessarily unital) ring R is called *Boolean* if $x^2 = x$ whenever $x \in R$.

1. Show that every Boolen ring is commutative.
2. Show that every finitely generated ideal in a Boolean ring is principal.

Modules and linear algebra theory

MLA - I: Let F be a field and consider the direct sum of cyclic $F[x]$-modules

$$V = \frac{F[x]}{(x+1)^2} \oplus \frac{F[x]}{(x^2-1)} \oplus \frac{F[x]}{(x-1)^2}.$$

1. Determine the invariant factors and the elementary divisors of V as an $F[x]$-module.
2. Determine the rational canonical form of the linear transformation $T : V \to V$ given by multiplication by x.

Make sure you include all the details including stating clearly all results you use in your proof!

MLA - II: Let $A, B \in M_n(\mathbb{R})$ be matrices satisfying $AB - BA = A$. Show that A is nilpotent.

MLA - III: Let R be a commutative Noetherian ring with identity and let M be a finitely generated R-module. Assume that $f : M \to M$ is a surjective R-homomorphism. Show that f is an isomorphism. Include all details and state clearly all results you use in your answer!

MLA - IV: Let Q be an R-module. Assume that for every left ideal I of R, any R-module homomorphism $f : I \to Q$ can be extended to an R-module homomorphism $\tilde{f} : R \to Q$. Show that Q is an injective R-module.
For the proof you are not allowed to just quote a theorem but instead you have to show all details!

Fields and Galois theory

F - I: Solve at your choice ONE of the following problems:

1. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is a Galois extension of degree 4 with cyclic Galois group.

   Make sure you include all the details including stating clearly all results you use in your proof!
2. Compute the splitting field of $x^3 - 2$ over $\mathbb{Q}$ and describe its Galois group over $\mathbb{Q}$.

   Make sure you include all the details including stating clearly all results you use in your proof!

F - II: Let G be a finite group. Show that there exists a fields extension $F \subset E$ such that E/F is Galois and $Gal(E/F)$ is isomorphic to G. Include all details and state clearly all results you use in your proof!

F - III:

1. Define the notion of character of a group with values in a field and then explain the notion of linear independence for such maps.
2. Show that any finite collection of distinct characters is linearly independent.

F - IV: Let $F \subset F$ be a finite extension of fields. Show that $F \subset E$ is simple if and only if there are only finitely many intermediate subfields of E containing F. Make sure you include all details in your proof!

Answers

Your Choice

G-

Solution:

Your Choice

G-

Solution:

Your Choice

R-

Solution:

Your Choice

R-

Solution:

Your Choice

MLA-

Solution:

Your Choice

MLA-

Solution:

Your Choice

F-

Solution:

Your Choice

F-

Solution:

Scratch Paper

Scratch Paper

Scratch Paper

Scratch Paper