Math@Iowa Analysis Qualifying Exam
Friday, August 23, 2024
9:00 am - 12:00 noon in Room 118 MLH

Examination Committee: Professors Raúl E. Curto and Palle E.T. Jørgensen

Your Name:

Instructions

1. This is a closed-book exam; no books or notes are allowed. This exam includes a Prep Sheet (attached), which you may use as a reference.
2. You are not allowed to use any internet resources; use of calculators is not allowed.
3. This Qualifying Exam consists of 10 questions, broken down into two parts of 5 questions each, for a maximum score of 70 points. Please submit **a total of seven question, with at least three chosen from each Part**. Please summarize your choices in the table below, by placing a ✓ inside the box located next to each choice.

5. Please write your answers within the space provided for each question. If at all possible, please write your answers in ink (blue, black or red). However, if writing math using a pencil is what you always do, by all means use a pencil for this exam too.
6. **For each solution you provide, make sure to include all the details, and state correctly all the known results you use. If appropriate, feel free to refer to the attached Prep Sheet.**
7. A score in the range **63 − 70** guarantees a **Ph.D. Pass**; a score in the range **56 − 62** guarantees an **M.S. Conditional Pass**.
8. **Before you start, please write your name on the space provided in this page. When you finish, please turn in this exam booklet.**

☐ R-1A	☐ R-1B		☐ C-1A	☐ C-1B
☐ R-2			☐ C-2A	☐ C-2B
☐ R-3A	☐ R-3B		☐ C-3	
☐ R-4			☐ C-4A	☐ C-4B
☐ R-5A	☐ R-5B		☐ C-5A	☐ C-5B

PART I: Real Analysis

(10 points) **(R-1)**

Choose either Problem R-1A or Problem R-1B below and solve it.

Problem R-1A. Let K be the set obtained by intersecting the Cantor set $C \subseteq [0,1]$ with the closed interval $[\frac{1}{3}, \frac{5}{6}]$. Prove that K and C are equipotent, by finding an explicit formula for a bijective function $f : K \to C$. Your answer must include a **detailed proof of the injectivity and surjectivity** of f.

Problem R-1B. Determine if the following statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Consider the sequence of real-valued functions

$$f_n(x) := e^{-2x} - ne^{-nx} \qquad (x \geq e;\ n \geq 3)$$

together with the function

$$f(x) := e^{-2x} \qquad (x \geq e).$$

(True or False) Fatou's Lemma holds for the sequence $\{f_n\}$ and the function f.

Problem R-1 cont.

(10 points) **(R-2)** Let $f \in L^1([0,1])$. A point $x \in (0,1)$ is said to be a *Lebesgue point* of f if

$$\lim_{h \to 0_+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)| \, dt = 0.$$

Consider now the Dirichlet function $f \in L^1([0,1])$ given as $\chi_{\mathbb{Q} \cap [0,1]}$; that is, f is the characteristic function of the set of rational numbers in $[0,1]$. Find the set of Lebesgue points of $1 - f$.

Problem R-2 cont.

(10 points) **(R-3)**

Choose either Problem R-3A or Problem R-3B below and solve it.

Problem R-3A. For $n \geq 1$, consider the vector space $\mathbb{C}^n$. For a vector $\mathbf{v} \equiv (v_1, \ldots, v_n) \in \mathbb{C}^n$, let

$$\|v\|_2 := \left(\sum_{k=1}^{n} v_k^2 \right)^{1/2}$$

and

$$\|v\|_4 := \left(\sum_{k=1}^{n} v_k^4 \right)^{1/4}$$

be the 2–norm of $\mathbf{v}$ (also known as the Euclidean norm of $\mathbf{v}$) and the 4–norm of $\mathbf{v}$, respectively. Prove that, for all vectors $\mathbf{v} \in \mathbb{C}$, the following chain of inequalities holds:

$$\|v\|_4 \leq \|v\|_2 \leq n^{1/4} \|v\|_4 .$$

Problem R-3B. Let $a, b \in \mathbb{R}$, and let f be an absolutely continuous function on $[a, b]$. Assume that there exists a constant $M > 0$ such that

$$|f'| \leq M$$

almost everywhere on $[a, b]$. Prove that f is Lipschitz on $[a, b]$.

Problem R-3 cont.

(10 points) **(R-4)** Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < +\infty$, where m denotes Lebesgue measure. For $1 \leq p < q \leq \infty$, it is well known that $L^q(E) \subseteq L^p(E)$. In particular, $L^2(E) \subseteq L^1(E)$. On the other hand, the spaces of sequences ℓ^2 and ℓ^1 satisfy the reverse inclusion; that is, $\ell^1 \subseteq \ell^2$.

(i) Find a concrete example of a measurable function f such that $f \in L^1([-1,1]) \setminus L^2([-1,1])$.

(ii) Find a concrete example of a sequence $x \equiv \{x_n\}_{n=1}^{\infty}$ such that $x \in \ell^2 \setminus \ell^1$.

Problem R-4 cont.

(10 points) **(R-5)**

Choose either Problem R-5A or Problem R-5B below and solve it.

Problem R-5A. Determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Let $\{E_k\}$ be a **countable** collection of **disjoint Lebesgue measurable** subsets of $\mathbb{R}$.

(True or False) For any set A in $\mathbb{R}$,

$$m^*[A \cap (\bigcup_k E_k)] = \sum_k m(A \cap E_k),$$

where m^* denotes the **Lebesgue outer measure**.

Problem R-5B. Let E be a measurable subset of $\mathbb{R}$, let $f \in L^1(E)$, and let $\varepsilon > 0$ be given. Prove that there is a measurable set $A \subseteq E$ with $m(A) < \infty$ such that

(a) $\sup_{x \in A} |f(x)| < \infty$

and

(b) $\int_{A^c} |f| \leq \varepsilon$.

Problem R-5 cont.

PART II: Complex Analysis

(10 points) **(C-1)**
Choose either Problem C-1A or Problem C-1 below and solve it.

Problem C-1A. On the space $H(\mathbb{D})$ of analytic functions on the open unit disc, consider the Hilbert and Cesàro transforms, defined (respectively) as

$$H(f)(z) := \int_0^1 \frac{f(t)}{1-tz}\,dt \quad (f \in H(\mathbb{D}), z \in \mathbb{D})$$

and

$$C(f)(z) := \frac{1}{z} \cdot \int_0^z \frac{f(t)}{1-t}\,dt \quad (f \in H(\mathbb{D}), 0 \neq z \in \mathbb{D}).$$

(a) Given the sequence of functions $e_n(z) := z^n \quad (z \in \mathbb{D})$, where n is a positive integer, prove that

$$C(e_n) = e_n H(e_n) \quad (\text{for all } n \geq 1).$$

(b) Use the above identity to show that for every $n \geq 1$, the function $C(e_n)$ has a removable singularity at $z = 0$.

Problem C-1B. Determine if each of the given statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

(i) Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$.
(True or False) There exists an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f'(\frac{1}{2}) = 2$.

(ii) Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, and let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function. Assume that f has two distinct fixed points in $\mathbb{D}$.
(True or False) f is the identity function; that is, $f(z) = z$ for every $z \in \mathbb{D}$.

Problem C-1 cont.

(10 points) **(C-2)**

Choose either Problem C-2A or Problem C-2B below and solve it.

In Problems C-2A and C-2B, determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Problem C-2A. Recall that a nonempty subset G of $\mathbb{C}$ is called *star-shaped* if there is a point a in G such that for every point z in G, the line segment $[a, z]$ lies entirely in G.

Let G be an open, star-shaped subset of $\mathbb{C}$, and assume that $G \neq \mathbb{C}$.

(True or False)

Then G is **conformally equivalent** to $\mathbb{D}$, where $\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$; that is, there exists an analytic function $f : G \to \mathbb{D}$ such that f is injective and surjective.

Problem C-2B. Let $B(1; 1)$ denote disk of radius $r = 1$ centered at $a = 1$. Let $u : B(1; 1) \to \mathbb{R}$ be given by

$$u(z) := \mathrm{Re}(\sqrt{z} - \sqrt{z+i}) \qquad (z \in B(1; 1)).$$

(True or False) u is harmonic in $B(1; 1)$.

Problem C-2 cont.

(10 points) **(C-3)** **Use complex analysis methods** to show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4}\, dx = \frac{\pi}{e^2}.$$

Problem C-3 cont.

(10 points) **(C-4)**

Choose either Problem C-4A or Problem C-4B below and solve it. In Problems C-4A, determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Problem C-4A. On the region $G := \mathbb{C} \setminus (-\infty, 0]$, we define $f(z) = z^z$, as follows:

$$f(z) := \exp(z \log z) \qquad (z \in G).$$

(i) Calculate $f(i)$ and express your answer in either Cartesian or polar form.

(ii) (True or False) f establishes a **conformal equivalence** between G and $\mathbb{C} \setminus \{0\}$; that is, f is analytic, injective, and surjective.

Problem C-4B. In $\mathbb{C}$, consider the polynomial

$$P(z) = z^5 + 3z + 1.$$

Prove that:

(i) P has exactly one zero in the open unit disk $\mathbb{D}$; and

(ii) the remaining four zeros are in the annulus $\text{ann}(0; 1, \sqrt{2}) := \{z \in \mathbb{C} : 1 < |z| < \sqrt{2}\}$.

Problem C-4 cont.

(10 points) **(C-5)**

Choose either Problem C-5A or Problem C-5B below and solve it.

Problem C-5A. Determine if the following statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Let $0 \neq z \equiv x + iy \in \mathbb{C}$.
(True or False). Then

$$\left| \sum_{n=0}^{\infty} \frac{z^n}{n!} \right| \leq \left| \sum_{n=0}^{\infty} \frac{x^n}{n!} \right| .$$

Problem C-5B. Let

$$f(z) = \frac{1}{z(z+1)^2} \quad (z \neq 0, z \neq -1).$$

Find the Laurent Expansion of f, in powers of z, in the annulus

$$\mathrm{ann}(0;\, 1, \infty) := \{z \ \in \ \mathbb{C} : \ 1 \ < \ |z| \ < \ +\infty\}.$$

Problem C-5 cont.

Additional Space

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Additional Space

End of Exam