Ph.D. Qualifying Exam in Topology

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Instructions.

- Do eight problems: four from part A and four from part B.
- This is a closed book examination: you should have no books, technology or paper of your own. Paper will be provided by the test center.
- Please do your work on the paper provided according to the format outlined below.
  - On each page of your solutions
    - Write your name
    - Write the page number
    - Indicate which problem is being addressed
  - When you start a new problem, start a new page
  - Only write on one side of the paper
  - Make a cover page and indicate which eight problems you want graded.
- Always justify your answers unless explicitly instructed otherwise.
- You may use theorems if the problem is not a step in proving that theorem. You must to state any theorems that you use clearly and carefully.

Part A - Algebraic Topology

In the problems below, the symbols for disk D^n and sphere S^n can be understood to mean the subspaces below

$$D^n := \{v \in \mathbb{R}^n : |v| \leq 1\} \qquad S^n := \{v \in \mathbb{R}^{n+1} : |v| = 1\}$$

1. State and prove the 2-dimensional Brouwer fixed point theorem.
2. Prove that if a path-connected locally path-connected space X has $\pi_1(X)$ finite then every map $X \to S^1$ is nulhomotopic.
3. Prove that S^∞ is contractible.
4. (a) What assumptions about a space Y can be made which ensure that there exists a space X for which Y is the universal cover?
   (b) Construct an example of a space Y and prove that there is no space X such that Y is the universal cover of X.
   (c) What assumptions about a space X can be made which ensure that there exists a space Y which is the universal cover of X?
   (d) Construct an example of a space X and prove that there is no space Y such that Y is the universal cover of X.
5. Suppose G is a topological group. In more detail, G is a set which is both a group (G, m, i) with multiplication $m : G \times G \to G$ and inverse $i : G \to G$ and a topological space in such a way that the group structure maps m and i are continuous. Let $1 \in G$ denote the identity element of G. Prove that the fundamental group $\pi_1(G, 1)$ of G is an abelian group.
6. (a) Use Seifert Van-Kampen theorem to compute $\pi_1(M)$ where M is a quotient of the union of two solid tori

   $$M := (S^1 \times D^2) \sqcup (S^1 \times D^2)/\sim_{(a)}$$

   where $(x, y) \sim_{(a)} (x', y')$ when both points are contained in the boundary $(x, y), (x', y') \in \partial(S^1 \times D^2)$ and one is the inversion of the other $(x, y) = (y', x')$.

(b) Use Seifert Van-Kampen theorem to compute $\pi_1(N)$ where N is a quotient of the union of two solid tori

$$N := (S^1 \times D^2) \sqcup (S^1 \times D^2) / \sim_{(b)}$$

where $(x, y) \sim_{(a)} (x', y')$ when both points are contained in the boundary $(x, y), (x', y') \in \partial(S^1 \times D^2)$ and they are equal $(x, y) = (x', y')$.

Part B

1. Prove that $\mathrm{SL}(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det(A) = 1\}$ is a manifold and find its dimension.

2. Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a smooth embedding. Compute the integral $\int_\gamma \theta$ when
$$\theta = xy^2dx + x^2ydy$$

3. Suppose that N is a closed submanifold of the smooth manifold M, and $f\colon N \longrightarrow \mathbb{R}$ is a smooth function. Shown that f can be extended to a smooth function on M.

4. Show that
$$f\colon \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}^5, \quad f(x,y,z) = (xy, yz, zx, x^2 - y^2, x^2 + y^2 + z^2 - 1)$$
is an immersion. Use f to construct an embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{R}^4$.

5. Show that any mobius function
$$f(z) = \frac{az+b}{cz+d}\colon \mathbb{C} - \{-d/c\} \longrightarrow \mathbb{C}$$
extends to a holomorphic diffeomorphism of $\mathbb{CP}^1$.

6. Let ψ be the 2-form on $\mathbb{R}^3 - \{0\}$ defined by
$$\psi = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}.$$
Let $\Sigma \subset \mathbb{R}^3 - \{0\}$ be a smooth compact surface that is the boundary ∂U of a compact 3-manifold-with-boundary $U \subset \mathbb{R}^3$. Let's agree to give the "bounded domain" U the orientation it inherits from $\mathbb{R}^3$, and then use this to induce the corresponding "out-pointing" boundary orientation on $\Sigma = \partial U$. Prove that
$$\frac{1}{4\pi}\int_\Sigma \psi = \begin{cases} 1 & \text{if} \quad 0 \in U, \\ 0 & \quad \text{otherwise.} \end{cases}$$