Ph.D. qual. exam./M.S. comp. exam. on Numerical analysis.
Monday August 19, 2024, 9am.

There are two Parts. **In Part I, answer at least 5 out of 7 questions. In Part II, answer at least 5 out of 7 questions.** Only the best 5 of each Part will count. Always show your calculations and justify your answers. Please **return this question sheet** at the end of the examination.

Part I, answer at least 5 out of 7 questions:

1. Give (by hand) at least 10 correct significant digits in base 10 of the mathematical expression (using radians!)
$$\frac{\sin(0.0003) - 0.0003}{10^{-6}}.$$
Explain why it is not recommended to evaluate directly this expression on a calculator. Note: you are not asked to prove that your answer has 10 correct significant digits in base 10.

2. For $n = 3$ give the Lagrange polynomials $\ell_0(x), \ell_1(x), \ell_2(x), \ell_3(x)$ (do not "simplify") corresponding to the values $x_0 = -1$, $x_1 = 0$, $x_2 = 2$, $x_3 = 3$. Then if possible find values $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that
$$\alpha_0\ell_0(x) + \alpha_1\ell_1(x) + \alpha_2\ell_2(x) + \alpha_3\ell_3(x) = 8 - 3x.$$
If not possible, explain why.

3. (a) Define what is a cubic spline. In particular what is a natural cubic spline?
   (b) We consider the following function on the interval $[0, 3]$
$$s(x) := \begin{cases} -4x^2 + 2x^3 & \text{for} \quad x \in [0, 1], \\ 2 - 6x + 2x^2 & \text{for} \quad x \in [1, 3]. \end{cases}$$
      i. Does $s(x)$ interpolate the 3 points $(x_0, y_0) = (0, 0)$, $(x_1, y_1) = (1, -2)$, $(x_2, y_2) = (3, 4)$?
      ii. Is $s(x)$ a cubic spline?
      iii. If $s(x)$ is a cubic spline, is it a natural cubic spline?

4. For $s = 3$ find the weights b_1, b_2, b_3 and nodes c_1, c_2, c_3 of a symmetric quadrature formula
$$h_j\left(b_1 f(x_j + c_1 h_j) + b_2 f(x_j + c_2 h_j) + b_3 f(x_j + c_3 h_j)\right) \approx \int_{x_j}^{x_j+h_j} f(x)dx$$
of highest possible order satisfying
$$c_1 = \frac{1}{5}.$$
What is its order?

5. Consider the periodic function $f(t) = t$ on the interval $[-\pi, \pi]$ repeated periodically on $\mathbb{R}$ with period 2π. Find the trigonometric polynomial
$$S_3(t) = \frac{a_0}{2} + a_1\cos(t) + b_1\sin(t)$$
minimizing $\int_0^{2\pi}(f(t) - S_3(t))^2 dt$.

6. Show that $q_3(x) = \frac{1}{8} + x^2$ is the continuous minimax polynomial approximation of degree at most 3 to the function $f(x) = |x|$ on the interval $[-1, 1]$.

7. Define Newton's method and the secant method to find a zero to a scalar nonlinear equation $g(x) = 0$ with $g : [a, b] \to \mathbb{R}$. Give an advantage of the secant method compared to Newton's method.

Part II, answer at least 5 out of 7 questions:

1. (a) In term of matrices, what is the Cholesky decomposition of a real symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$ (do not give the details on how to find it)?

   (b) Based on the Cholesky decomposition of a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$ propose a simple way to calculate the scalar quantity

$$v^T A^{-1} v$$

   for any vector $v \in \mathbb{R}^n$ using a minimal number of arithmetic operations.

2. Consider the matrix

$$A := \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 2}.$$

   Find $x \in \mathbb{R}^2$ minimizing $\|b - Ax\|_2$ for

$$b := \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \in \mathbb{R}^3.$$

3. Consider the system of linear equations

$$\begin{bmatrix} 8 & 4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -4 \end{bmatrix}.$$

   Starting from the initial vector

$$x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} := \begin{bmatrix} \pi \\ \sqrt{7} \end{bmatrix},$$

   do Gauss-Seidel iterations converge to the solution x^* of the system of linear equations, i.e., do we have $\lim_{k \to \infty} x^{(k)} = x^*$? Prove convergence or divergence (computing the first few iterates is not a proof). Same question for the system of linear equations

$$\begin{bmatrix} 4 & 8 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -4 \end{bmatrix}$$

   with same initial vector $x^{(0)}$.

4. Consider the following nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \cos(t x_1) \\ e^{-3t} x_1^2 \end{bmatrix}.$$

   Give the value $x_1 = (x_{11}, x_{12})$ after one step with stepsize $h = 0.2$ of the Taylor series method of order 2 starting from $x_0 = (1, 2)$ at $t_0 = 0$.

5. Consider a system of ODEs $\dot{x} = f(t, x)$ with initial conditions $x(t_0) = x_0$ and the following explicit Runge-Kutta method

$$\begin{aligned} X_1 &= x_0 \\ X_2 &= x_0 + h \frac{1}{3} f(t_0, X_1) \\ X_3 &= x_0 + h\left(-f(t_0, X_1) + 2f(t_0 + h/3, X_2)\right) \\ x_1 &= x_0 + h\left(\frac{3}{4} f(t_0 + h/3, X_2) + \frac{1}{4} f(t_0 + h, X_3)\right) \end{aligned}$$

   Find the local order p of this RK method, $x(t_0 + h) - x_1 = O(h^{p+1})$.

6. Consider a system of ODEs $\dot{x} = f(t, x)$ with initial conditions $x(t_0) = x_0$. Among all explicit linear 2-step methods the method (using the notation $f_j := f(t_j, x_j)$)

$$x_{n+1} + 4x_n - 5x_{n-1} = h\,(4f_n + 2f_{n-1})$$

has the highest local order, what is it? Is this method generally globally convergent?

7. Consider the QR algorithm with dynamic single shift

$T_0 := A$

$k := 0$

while termination criteria not satisfied do

    compute Q_{k+1} *and* R_{k+1} | $T_k - p_k I = Q_{k+1} R_{k+1}$ *(*QR *decomposition)*

    $T_{k+1} := R_{k+1} Q_{k+1} + p_k I$

    $k := k + 1$

end while

Show that if $A \in \mathbb{R}^{n\times n}$ is antisymmetric/skew-symmetric ($A^T = -A$) then all T_k for $k = 0, 1, 2, 3, \ldots$ are also antisymmetric/skew-symmetric (i.e., $T_k^T = -T_k$).