Name:____________________________________

Ph.D. Qualifying Exam in PDE

August 2024

This booklet contains five problems with equal weights on seven sheets of paper including this cover and the extra blank sheet on the back.

Please solve any four from these five problems in this PDE part of the exam.

You can use the back of the sheets and the extra blank one at the back.

Note to Proctor: Please print the exam one-sided on seven pages of paper.

1. a. Define Sobolev Space $H^1(B_1)$ where B_1 is the unit ball in the $n-$dimensional space; And define the weak derivatives for functions in $H^1(B_1)$;

   b. State Sobolev or Morrey embedding theorems for $H^1(B_1)$ for dimension $n = 1$ and $n = 3$; and write down the above embedding theorems as inequalities for functions in $H^1(B_1)$.

2. a. Suppose u is a harmonic function defined in the domain outside the unit disk: $B_1^C = \{(x, y) : x^2 + y^2 > 1\}$ and suppose that $|u| \leq 1$ with the boundary condition $u = 0$ on ∂B_1. Prove that $u \equiv 0$.

   b. Is the above conclusion still true in 3 dimensions? Prove your statement in this case.

3. Let $g \in L^2(\partial B_1)$. Prove the map $u \to \displaystyle\int_{\partial B_1} ugd\sigma$ defines a continuous linear functional on $H^1(B_1)$.

4. Let B_1 be the unit ball in $\mathbf{R}^n$. For $f \in L^2(B_1)$ and $g \in L^2(\partial B_1)$, define and prove there is a unique weak solution to boundary value problem

$$\begin{cases} -\Delta u &= f, \quad \text{in } B_1, \\ \frac{\partial u}{\partial \nu} + 2u &= g, \quad \text{on } \partial B_1. \end{cases}$$

5. Consider the heat equation on the whole x-axis. Show that there is a unique solution $u = u(x, t)$ with $0 \leq u(x, t) \leq e^{x+t}$ to

$$\begin{cases} u_t - u_{xx} &= 0, \quad x \in \mathbb{R}, \quad 0 < t < \infty, \\ u(x, 0) &= (e^x - 1)^+, \text{(the plus part)}. \end{cases}$$