Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

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Instructions:

- Do EXACTLY TWO problems from EACH of the four sections.
- Please start a new page for every new problem and put your name on each sheet.
- Please ONLY WRITE on ONE SIDE of each sheet.
- Justify your answers and show your work.
- Please write legibly.
- In answering any part of a question, you may assume the results in previous parts of the SAME question, even if you have not solved them.
- Please turn in the exam questions with your solutions.

Notations:

We adopt standard notations. Namely:

- We write $\mathbb{C}, \mathbb{R}$ and $\mathbb{Q}$ to denote the field of complex numbers, real numbers and rational numbers, respectively; we write $\mathbb{Z}$ to denote the ring of rational integers; when p is a prime number, we write $\mathbb{F}_p$ to denote the finite field with p elements.
- Throughout this exam, R denotes a ring with identity 1; R is called an integral domain if $1 \neq 0$ and R is commutative with no zero divisors.
- All R-modules are assumed to be unital left R-modules.

1 Groups

1. Consider the additive abelian group $\mathbb{Q}/\mathbb{Z}$.

   (a) (3 points) Prove that $\mathbb{Q}/\mathbb{Z}$ is a torsion group.

   (b) (6 points) Prove directly that

   $$\mathbb{Q}/\mathbb{Z} = \bigoplus_p (\mathbb{Q}/\mathbb{Z})(p),$$

   where the direct sum is taken over all primes, and $(\mathbb{Q}/\mathbb{Z})(p)$ consists of those elements which can be represented by rational numbers of the form a/p^k with $a \in \mathbb{Z}$ and k being a positive integer.

   You cannot use a general theorem which says that any torsion abelian group A is the direct sum of $A(p)$.

   (c) (6 points) Prove that for any positive integer n, there is a unique subgroup of $\mathbb{Q}/\mathbb{Z}$ which is of order n.

2. Let U and V be subgroups of a group G. Let u be a normal subgroup of U, and let v be a normal subgroup of V.

   (a) (3 points) Prove that $(u \cap V)v$ is normal in $(U \cap V)v$.

   (b) (3 points) Prove that $(u \cap V)(U \cap v)$ is normal in $U \cap V$.

   (c) (9 points) Prove that the two factor groups are isomorphic:

   $$(U \cap V)v/(u \cap V)v \approx (U \cap V)/(u \cap V)(U \cap v).$$

3. (15 points) Let G be a finite group and let p be the **smallest** prime number diving $|G|$. Prove that every subgroup H of G of index p is normal in G.

2 Rings

1. Let A be a principal ideal domain.

   (a) (9 points) Prove that every nonzero prime ideal in A is a maximal ideal.

   (b) (6 points) Prove that every nonzero proper ideal of A is a product of finitely many maximal ideals, uniquely determined up to order.

2. (15 points) Let k be a field. Consider the polynomial ring $k[x_1, \ldots, x_n]$ for $n \geq 1$. Determine for what $n \geq 1$ the polynomial ring $k[x_1, \ldots, x_n]$ is a principal ideal ring and for what $n \geq 1$ it is not a principal ideal ring. Prove your assertions.

3. Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$ be the set of polynomials in x with rational coefficients whose constant coefficient is an integer.

   (a) (6 points) Prove that R is an integral domain and its units are ± 1.

   (b) (9 points) Prove that the irreducible elements in R are either of the form $\pm p$, where p is a prime in $\mathbb{Z}$, or they are polynomials $f(x) \in R$ that are irreducible in $\mathbb{Q}[x]$ and have constant coefficient ± 1.

3 Linear Algebra and Module Theory

1. (15 points) Let R be a principal ideal domain and let E be a free module over R of finite rank. Let F be a submodule. Prove that F is free.

2. (15 points) Consider a commutative diagram of R-modules such that each row is exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

   Prove that if any two of f, g, and h are isomorphisms, then so is the third.

3. (15 points) Let $n \geq 2$ be an integer. Determine the Jordan canonical form of the $n \times n$ matrix over $\mathbb{Q}$ whose entries are all equal to 1 except that the entries along the main diagonal are equal to 0.

4 Field Theory and Galois Theory

1. Let m and n be positive integers. Consider subfields $\mathbb{F}_{p^m}$ and $\mathbb{F}_{p^n}$ in any algebraic closure of $\mathbb{F}_p$.

   (a) (6 points) Prove that $\mathbb{F}_{p^n}$ is contained in $\mathbb{F}_{p^m}$ if and only if n divides m.

   (b) (9 points) Assume this is the case with $m = nd$ for some $d \in \mathbb{Z}$. Prove that $\mathbb{F}_{p^m}$ is normal and separable over $\mathbb{F}_{p^n}$.

2. (15 points) Prove that $\mathbb{C} = \mathbb{R}(i)$ is algebraically closed. You may use the following properties of $\mathbb{R}$: $\mathbb{R}$ is an ordered field, every positive element is a square, and every polynomial of odd degree in $\mathbb{R}[X]$ has a root in $\mathbb{R}$.

3. (15 points) Let p be a prime number, and consider $f(x) = x^p - 2 \in \mathbb{Q}[x]$. Let G be the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$. Prove that G is isomorphic to the group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{F}_p) \,\middle|\, a, b \in \mathbb{F}_p, a \neq 0 \right\}.$$